Highly Connected Random Geometric Graphs

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Abstract

Let $\mathcal{P}$ be a Poisson process of intensity one in a square S_n of area n. We construct a random geometric graph $G_{n,k}$ by joining each point of $\mathcal{P}$ to its k nearest neighbours. For many applications it is desirable that $G_{n,k}$ is highly connected, that is, it remains connected even after the removal of a small number of its vertices. In this paper we relate the study of the s-connectivity of $G_{n,k}$ to our previous work on the connectivity of $G_{n,k}$. Roughly speaking, we show that for $s = o(\log n)$, the threshold (in k) for s-connectivity is asymptotically the same as that for connectivity, so that, as we increase k, $G_{n,k}$ becomes s-connected very shortly after it becomes connected.

1 Introduction

The following model was motivated by the study of wireless ad-hoc networks. Consider a Poisson process $\mathcal{P}$ of intensity one in a square S_n of area n (all our results will also apply for the case of n points uniformly distributed in a square). We define the random geometric graph $G_{n,k}$ by joining each point of $\mathcal{P}$ to its k nearest neighbours. Here, and throughout this paper, distance is measured using the Euclidean l_2 norm.

One can now ask various questions, for instance, how large should k be to guarantee the existence of a *giant component* in $G_{n,k}$, that is, one containing a positive proportion of the vertices as $n \to \infty$? Another area of interest is connectivity: how large should k be to guarantee the connectivity of $G_{n,k}$? This problem has been extensively studied in the context of wireless ad-hoc networks [5, 6, 7, 8, 9, 10]. Of course, the word "guarantee" is used probabilistically: a typical result will state that for some $k = f(n)$ the probability that $G_{n,k}$ is connected tends to one as $n \to \infty$. From now on, we shall use the phrase "with high probability" (**whp**) to mean "with probability tending to one as $n \to \infty$". Also, all logarithms in this paper are to the base e.

In [1] we prove that if $k \leq 0.3043 \log n$ then $G_{n,k}$ is not connected **whp**, while if $k \geq 0.5139 \log n$ then $G_{n,k}$ is connected **whp**. This greatly improved the earlier bounds due to Xue and Kumar [11] and Gonzáles-Barrios and Quiroz [4]. More recently, we proved [2] that there is a critical value c, such that for $c_1 < c$ and $k \leq c_1 \log n$, $G_{n,k}$ is not connected **whp**, and that for $c_2 > c$ and $k \geq c_2 \log n$, $G_{n,k}$ is connected **whp**. However, the value of this constant c remains unknown. Numerical results [1] indicate that it is close to the above lower bound, namely 0.3043.

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In the original context of wireless ad-hoc networks, each point of $\mathcal{P}$ is a radio transceiver, and we suppose that each such radio is able to establish a direct two-way connection with the k radios nearest to it. In addition, messages can be routed via intermediate radios, so that a message can be sent indirectly from radio S to radio T through a series of radios $S = S_1, S_2, \ldots, S_n = T$, each one having a direct connection to its predecessor. The above results show that if $k \geq 0.5139 \log n$ then, with high probability, any two radios can communicate with each other, either directly or indirectly.

In this paper we investigate the fault tolerance of these networks, in the following sense. Suppose that, once the network has been constructed, some of the radios are destroyed or develop faults, so that they can no longer transmit or receive messages. Is it still possible for every pair of remaining radios to communicate with each other, or were the removed radios essential for such communication? Ideally, we would like the network to remain connected, in this sense, for *any* choice of s removed radios, where s is small compared to n. This does not happen, for instance, if the network contains a few "hubs", through which an unusually high proportion of messages are routed, for then removing the hubs would disconnect the network.

To make this question precise, we introduce the standard graph theoretic notion of s-connectivity. A graph G is said to be s-connected if it contains at least $s+1$ vertices, and the removal of any $s-1$ of its vertices does not disconnect it. In this paper we show that for $s = o(\log n)$, $G_{n,k}$ becomes s-connected "just after" it becomes connected. Specifically, if $s = o(\log n)$ and c is such that $G_{n,\lfloor c \log n \rfloor}$ is connected **whp**, then, for all $\varepsilon > 0$, $G_{n,\lfloor (c+\varepsilon) \log n \rfloor}$ is s-connected **whp**. In particular, if c is the critical constant for connectivity described above, then c is also critical in the same sense for s-connectivity, for any $s = o(\log n)$. Broadly speaking, our results show that a connected wireless ad-hoc network, constructed as above, can be made very fault tolerant by increasing the number of connections only very slightly, a feature which is perhaps even more relevant in practice than mere connectivity.

Wireless ad-hoc networks are only one type of network: another is given by a network of television transmitters broadcasting over a certain region. Here, it is important that every point in the region falls within the range of some transmitter. In [1] we investigated the question of whether a network of n tranmitters in a square, each choosing its range so as to reach at least k other transmitters, would in fact cover the entire region in the above sense. More precisely, with $G_{n,k}$ as above, surround each vertex by the smallest disc containing its k nearest neighbours: we studied the values of k for which these discs cover the square S_n **whp**.

Once again, for practical applications, fault tolerance of such a transmitter network is frequently more of an issue than simply coverage: one can imagine that some transmitters are destroyed by enemy action, disabled by bad weather or simply need replacing. In this case, we would like the remaining transmitters to still cover the region in the sense described above. In this paper, we show that for $s = o(\log n)$, if c is such that for $k = \lfloor c \log n \rfloor$ the discs cover S_n **whp**, then, for all $\varepsilon > 0$, the discs obtained by taking $k' = \lfloor (c+\varepsilon) \log n \rfloor$ form an s-cover of S_n **whp**, that is, one in which each point of S_n lies in at least s of the discs. Thus, as before, one can make the network very fault tolerant by increasing the parameter k only very slightly.

All our results will apply not only for Poisson processes, but also for n points placed in a square of area n with the uniform distribution. Indeed, one can view our Poisson process as simply the result of placing X points in the square, where $X \sim \mathrm{Po}(n)$.

2 Results

Our main results concern s-connectivity. Theorem 1 deals with the case where s is a constant.

Theorem 1. *Fix $s \in \mathbb{N}$. Suppose $k = k(n)$ is such that $G_{n,k}$ is connected* **whp**. *Then, for any $\varepsilon > 0$, the graph $G_{n,\lfloor k(1+\varepsilon) \rfloor}$ is s-connected* **whp**.

Our second main result, Theorem 2, deals with the case $s = o(\log n)$.

Theorem 2. *Let $s = s(n) = o(\log n)$. Suppose c is such that $G_{n,\lfloor c \log n \rfloor}$ is connected* **whp**. *Then, for any $\varepsilon > 0$, $G_{n,\lfloor (c+\varepsilon) \log n \rfloor}$ is s-connected* **whp**.

Since this paper was written, we proved the existence of a critical constant c for connectivity in the k-nearest neighbour model [2], so that $G_{n,\lfloor c' \log n \rfloor}$ is connected **whp** if $c' > c$ and not connected **whp** if $c' < c$. Thus Theorem 2 implies that for any $c' > c$ and $s = o(\log n)$, $G_{n,\lfloor c' \log n \rfloor}$ is s-connected **whp**. It also implies Theorem 1, although we include the proof of Theorem 1 in this paper since it is easier than that of Theorem 2 and makes a good warm up exercise.

Our sharpest result is Theorem 10, which we do not state here owing to its somewhat complicated hypothesis.

The plan of the paper is as follows. Section 3 contains our results on connectivity. We need the main theorem of [1] and three technical lemmas before we can begin. These are followed by Lemma 7, which embodies the main idea relating s-connectivity to connectivity, and it together with Lemma 8 enables us to establish Theorem 1. For Theorem 2, we require a strengthened version of a sharpness result ("sharpness in n") from [1], and for Theorem 10 we need to prove a new sharpness result ("sharpness in k") which we believe is of considerable interest in its own right. Section 4 contains analogous results for coverage, and we conclude with some open problems in Section 5.

3 s-Connectivity

We will require some slightly strengthened versions of theorems and lemmas from our earlier paper [1]. In each case the proof is an easy modification of the proof of the weaker counterpart in [1]. Throughout the paper, "diameter" will always mean Euclidean, and not graph, diameter.

Theorem 3 (Theorem 1 of [1])**.** *If $c \le 0.3043$ then $\mathbb{P}(G_{n,\lfloor c \log n \rfloor}$ is connected$) \to 0$ as $n \to \infty$. If $c > 1/\log 7 \approx 0.5139$ then $\mathbb{P}(G_{n,\lfloor c \log n \rfloor}$ is connected$) \to 1$ as $n \to \infty$.*

Lemma 4. *For fixed $c > 0$ and K, there exists $c' = c'(c, K) > 0$ such that, for any $k \ge c \log n$, the probability that $G_{n,k}$ contains two components each of diameter at least $c'\sqrt{\log n}$, or any edge of length at least $c'\sqrt{\log n}$ is $O(n^{-K})$.*

Proof. Immediate from the proofs of Lemmas 6 and 2 of [1]. ☐

Lemma 5. *For fixed $c' > 0$ and any $k \ge 0.3 \log n$, the probability that there exists a component of $G_{n,k}$ with diameter less than $2c'\sqrt{\log n}$, any of whose points lie closer than distance $c'\sqrt{\log n}$ from two sides of S_n, is $o(n^{-1/4})$.*

Proof. In the proof of Theorem 7 in [1] we show that the probability of the existence of such a component is $n^{o(1)} 3^{-k} \le n^{o(1)} e^{-0.3 \log 3 \log n} = o(n^{-1/4})$ as required. ☐

Lemma 6. *For any $k = k(n) \ge 0.7 \log n$ the probability that $G_{n,k}$ is not connected is $o(n^{-1/4})$.*

Proof. Again by the proof of Theorem 7 in [1] we see that the probability of a small component (one of diameter at most $c'\sqrt{\log n}$) near to no side of S_n is $n^{1+o(1)} 7^{-k} \le n^{1+o(1)} e^{-0.7 \log 7 \log n} = o(n^{-1/4})$, and that the probability of a small component near to exactly one side of S_n is $n^{1/2+o(1)} 5^{-k} \le n^{1/2+o(1)} e^{-0.7 \log 5 \log n} = o(n^{-1/4})$. Combining this with Lemma 4 and Lemma 5, the result follows. ☐

The following crucial lemma allows us to relate s-connectivity to connectivity at slightly smaller values of n and k. It is the main tool which enables us to prove Theorem 1, Theorem 2 and Theorem 10. Recall that n need not be an integer, since it is only the expected number of points in the square.

Lemma 7. *For any $s, d, k, n \in \mathbb{N}$ and $\delta = \delta(n) \in (0, 1)$ with $0 < d \leq k$ and*

$$\mathbb{P}(G_{n,k} \textit{ is not } s\textit{-connected}) \geq \theta,$$

we have

$$\mathbb{P}(G_{n(1-\delta),k-d+1} \textit{ is not connected}) \geq \theta\delta^{s-1}(1-\delta)^2 - n(ek\delta/d)^d.$$

Proof. Write $k' = k - d + 1$ and $n' = (1 - \delta)n$. Let $\mathcal{P}_n$ be a Poisson process of intensity one in a square S_n of area n. We may consider $\mathcal{P}_n$ as the union $\mathcal{P}_n = \mathcal{P}_{n'} \cup \mathcal{P}_{\delta n}$, where $\mathcal{P}_{n'}$ and $\mathcal{P}_{\delta n}$ are independent Poisson processes in S_n of intensities $1 - \delta$ and δ respectively.

Let $G = G_{n,k}$ be the graph obtained from $\mathcal{P}_n$ by joining each point of $\mathcal{P}_n$ to its k nearest neighbours. Let G' be the graph obtained from $\mathcal{P}_{n'}$ by joining each point of $\mathcal{P}_{n'}$ to its k' nearest neighbours. We wish to give a lower bound on the probability that G' is not connected. We do this in two stages. First we bound (from below) the probability that $G'' = G \setminus \mathcal{P}_{\delta n}$ is not connected, and second we bound (from above) the probability that G' is not a subgraph of G''. Note that $V(G') = V(G'') = \mathcal{P}_{n'}$ and that we are simply interested in the probability that $E(G') \subset E(G'')$ – we do not require that G' is an induced subgraph of G''.

Suppose that G is not s-connected: we know that this happens with probability at least θ. Then there is a set S of (at most) $s-1$ vertices whose removal disconnects G. Let x and y be two vertices not joined by a path in $G \setminus S$. Now if $S \subset \mathcal{P}_{\delta n}$ and $\{x, y\} \subset \mathcal{P}_{n'}$, then G'' will not be connected. This is because G'' will contain none of the vertices of S, so that x and y, both of which will lie in $V(G'') = \mathcal{P}_{n'}$, will not be connected by a path. The first event occurs with probability δ^{s-1}, and the second with probability $(1 - \delta)^2$. Thus

$$\mathbb{P}(G'' \text{ is not connected} \mid G \text{ is not } s\text{-connected}) \geq \delta^{s-1}(1-\delta)^2.$$

(Strictly speaking, G might not be s-connected because $\mathcal{P}_n$ contains fewer than $s + 1$ points, but then $|V(G'')| = |\mathcal{P}_{n'}| \leq 1$ with probability at least δ^{s-1}, so the above inequality still holds if we regard both the empty graph and an isolated vertex as not being connected.) Consequently,

$$\mathbb{P}(G'' \text{ is not connected}) \geq \theta\delta^{s-1}(1-\delta)^2.$$

A vertex $v \in \mathcal{P}_{n'} = V(G'')$ was originally joined to its k nearest neighbours in G. However, some of these neighbours might have belonged to $\mathcal{P}_{\delta n}$, so that v will not necessarily be joined to its k' nearest neighbours in G''. Nevertheless, if for each v, fewer than d of the k nearest neighbours of v in $\mathcal{P}_n = V(G)$ lie in $\mathcal{P}_{\delta n}$, then each vertex of G'' will be joined to at least its k' nearest neighbours in G'', which says precisely that G' is a subgraph of G''.

The probability that a specified subset of size d of the neighbours of a vertex $v \in V(G)$ all lie in $\mathcal{P}_{\delta n}$ is δ^d. Since there are $\binom{k}{d}$ such subsets, the probability that v is joined to at least d vertices of $\mathcal{P}_{\delta n}$ is at most $\binom{k}{d}\delta^d \leq (ek\delta/d)^d$. The expected number of vertices that are joined to at least d vertices of $\mathcal{P}_{\delta n}$ is therefore at most $n(ek\delta/d)^d$. This is an upper bound on the probability that there is such a vertex, and so it is an upper bound on the probability that G' is not a subgraph of G''.

Putting the pieces together, we see that

$$\begin{aligned}\mathbb{P}(G' \text{ is not connected}) &\geq \mathbb{P}(G'' \text{ is not connected}) - \mathbb{P}(G' \text{ is not a subgraph of } G'') \\ &\geq \theta\delta^{s-1}(1-\delta)^2 - n(ek\delta/d)^d\end{aligned}$$

as required. □

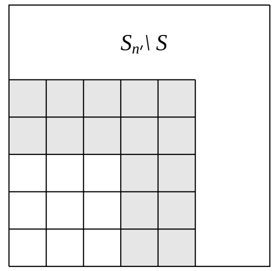

Figure 1: The squares S and $S_{n'}$ in the proof of Lemma 8.

The next lemma says that the probability that $G_{n,k}$ is connected is almost monotonically decreasing in n.

Lemma 8. *Suppose that n and $k(n) \geq 0.3 \log n$ are such that for all n*

$$\mathbb{P}(G_{n,k(n)} \text{ is not connected}) > p(n)$$

for some function $p(n) = \Omega(n^{-1/4})$. Then, for any $n' > n$ and any K,

$$\mathbb{P}(G_{n',k(n)} \text{ is not connected}) > p(n)/4 - O(n^{-K}).$$

Proof. Fix n' and write $k = k(n)$. Consider the square $S \subset S_{n'}$ of area n in the bottom left hand corner of $S_{n'}$. Let G be the k nearest neighbour graph formed by the points in S. The induced subgraph H of $G_{n',k}$ formed by the vertices in S is a subgraph of G. By hypothesis, with probability at least $p = p(n)$ the graph G is not connected. By Lemma 4 and the hypothesis of the theorem the probability that G has a component of diameter at most $c'\sqrt{\log n}$ is at least $p - O(n^{-K})$. Also, with probability at least $p/4 - O(n^{-K})$ it contains a small component (one of diameter at most $c'\sqrt{\log n}$) with a vertex in the bottom left hand quarter of S (by symmetry), and thus with probability at least $p/4 - O(n^{-K})$ a component F with no vertex within $\sqrt{n}/2 - c'\sqrt{\log n} > 0.4\sqrt{n}$ of $S_{n'} \setminus S$.

Divide the square S into 25 small squares. By Lemma 6 we may assume $k \leq 0.7 \log n$. The expected number of points in each of the small squares is $n/25$, and the probability that one such square has exactly $\ell \leq k$ points is $e^{-n/25}(n/25)^\ell/\ell!$. Therefore with probability $1 - O(e^{-n/25}(n/25)^{\log n}) \geq 1 - O(n^{-K})$, the top 10 and right 10 squares (a total of 16 squares) each contain at least k points. In this case, there will be no edge from any point in the bottom left 9 squares to $S_{n'} \setminus S$. The probability that this happens and that a small component F as above occurs is at least $p/4 - O(n^{-K}) - O(n^{-K})$, and, in this case, since H is a subgraph of G, F is a component in the original graph. ∎

Theorem 1, stated in Section 2, is the first of our results showing that s-connectivity occurs "shortly" after connectivity.

Proof of Theorem 1. Fix $\varepsilon > 0$ and let $k' = \lfloor k(1+\varepsilon) \rfloor$. Suppose that it is not true that $G_{n,k'}$ is s-connected **whp**. Then there exists $\theta > 0$ such that

$$\mathbb{P}(G_{n,k'} \text{ is not } s\text{-connected}) \geq \theta$$

for infinitely many n, say for $n \in \mathcal{N}$. The hypothesis implies that $k \geq 0.3 \log n$, and since $d = k' - k + 1 > k\varepsilon$, we have

$$n \left(\frac{ek'\delta}{d} \right)^d < n \left(\frac{ek'\delta}{k\varepsilon} \right)^{k\varepsilon} \leq n \left(\frac{e\delta(1+\varepsilon)}{\varepsilon} \right)^{k\varepsilon} = n \cdot e^{k\varepsilon \log(e\delta(1+\varepsilon)/\varepsilon)} \leq n^{1+0.3\varepsilon \log(e\delta(1+\varepsilon)/\varepsilon)},$$

which is $o(1)$ for some sufficiently small constant $\delta > 0$, depending on ε but not on n. It follows from Lemma 7 that for $n \in \mathcal{N}$

$$\mathbb{P}(G_{n(1-\delta),k} \text{ is not connected}) \geq \theta\delta^{s-1}(1-\delta)^2 - o(1).$$

Finally, by Lemma 8 (monotonicity)

$$\mathbb{P}(G_{n,k} \text{ is not connected}) \geq \tfrac{\theta}{4}\delta^{s-1}(1-\delta)^2 - o(1) = \Omega(1),$$

for $n \in \mathcal{N}$, which is a contradiction. □

We now extend Theorem 1 to the case $s = o(\log n)$. We need a strengthened version of Theorem 9 from [1]. It says that, for a fixed k, the probability of connectedness decays very sharply for n around its critical value.

Lemma 9. *Suppose that $k = k(n)$ and $p = p(n)$ are such that*

$$\mathbb{P}(G_{n,k} \textit{ is not connected}) > p = \Omega(n^{-1/4}). \tag{1}$$

Then

$$\mathbb{P}(G_{n',k} \textit{ is not connected}) > 1 - 1/e - o(1)$$

where $n' = (\lceil 2/p \rceil + 1)^2 n$.

Proof. First note that if $k < 0.3 \log n$ then $k < 0.3 \log n'$ and, thus, by Theorem 3 $G_{n',k}$ is not connected **whp**, and the lemma is trivially true. Thus we may assume $k \geq 0.3 \log n$. Let c' be the constant from Lemma 4 corresponding to $c = 0.3$ and some $K > 1/4$.

We say that a point $x \in V(G_{n,k})$ is *close* to a side s of S_n if the distance from x to s is less than $c'\sqrt{\log n'}$, and call a component G' of $G_{n,k}$ close to s if it contains points which are close to s. Further, we say that $x \in V(G_{n,k})$ is *central* if it is not close to any side s of S_n, and call a component G' of $G_{n,k}$ central if it consists entirely of central points. Finally, we call a component G' of $G_{n,k}$ *small* if it has diameter at most $c'\sqrt{\log n'}$, and *large* otherwise. Note that $c'\sqrt{\log n'} < 2c'\sqrt{\log n}$ for large n.

By (1) and Lemma 4, provided n is large enough, with probability more than $\frac{3}{4}p$, $G_{n,k}$ contains a small component, which can be close to at most two sides of S_n. Write α for the probability that we have a small central component of $G_{n,k}$. Write β for the probability that we have a small component of $G_{n,k}$ which is close to exactly one side of S_n, and γ for the probability that we have a component of $G_{n,k}$ close to two sides of S_n (so that it lies at a corner of S_n). We have $\alpha+\beta+\gamma > \frac{3}{4}p$, and, by Lemma 5, $\gamma = o(n^{-1/4})$ so we may assume that either $\alpha > \frac{p}{8}$ or $\beta > \frac{p}{2}$ provided that n is large enough. If we specify one side s of S_n, the probability that we obtain either a small central component or one which is close only to s is thus at least $\frac{p}{8}$.

Let $M = \lceil 2/p \rceil + 1$. We consider the larger square $S_{M^2 n} = S_{n'}$, and tessellate it with copies of S_n. We only consider the small squares of the tessellation incident with the boundary of $S_{M^2 n}$. Considering sides of these copies of S_n lying on the boundary of $S_{M^2 n}$, we see that we have $4(M-1)$ independent opportunities to obtain a small component G' in one of the small squares S, in such a way that G' can only be close to the boundary of S where that boundary is also part of the boundary of $S_{M^2 n}$. Such a component will also be isolated in $G_{M^2 n,k}$, since, by Lemma 4, **whp** no edge of $G_{M^2 n,k}$ has length greater than $c'\sqrt{\log n'}$. Therefore, if p' is the probability that $G_{M^2 n,k}$ is not connected, we have

$$p' \geq 1 - \left(1 - \tfrac{p}{8}\right)^{4(M-1)} - o(1) > 1 - e^{-\frac{p}{2}(M-1)} - o(1) \geq 1 - 1/e - o(1).$$

□

We can now prove Theorem 2, a version of Theorem 1 with a slightly stronger hypothesis and a much stronger conclusion.

Proof of Theorem 2. Fix $\varepsilon > 0$, let c be as in the statement of the theorem, and let $c' = c + \varepsilon$. Further, let $c'' = c + \varepsilon/2$, $\delta = \frac{\varepsilon}{2}e^{-4/\varepsilon - 1}$, $n' = (1-\delta)n$, $k = \lfloor c \log n \rfloor$, $k' = \lfloor c' \log n \rfloor$ and $k'' = \lfloor c'' \log n \rfloor$.

The bounds given by Theorem 3 show that the hypothesis of the theorem is not satisfied if $c \le 0.3$ and is satisfied if $c \ge 0.6$. Since the conclusion of the theorem for c and ε implies the conclusion for all larger c and ε it is sufficient to prove the theorem for $0.3 < c < c' < 1$.

Suppose the theorem is false. Then there is some $\theta > 0$ for which

$$\mathbb{P}(G_{n,k'} \text{ is not } s\text{-connected}) \ge \theta$$

for infinitely many n. Setting $d = k' - k'' + 1$ and noting that $k' < \log n$ and $d \ge \frac{\varepsilon}{2} \log n$, we have

$$\frac{ek'\delta}{d} < \frac{e(\log n)\frac{\varepsilon}{2}e^{-4/\varepsilon-1}}{\frac{\varepsilon}{2}\log n} = e^{-4/\varepsilon},$$

so

$$n(ek'\delta/d)^d < ne^{-4d/\varepsilon} \le ne^{-2\log n} = 1/n.$$

Thus by Lemma 7,

$$\mathbb{P}(G_{n',k''} \text{ is not connected}) \ge \theta\delta^{s-1}(1-\delta)^2 - 1/n = p(n).$$

Since $\delta < 1$ and $\theta > 0$ are constant and $s = o(\log n)$,

$$p(n) = \theta(1-\delta)^2 n^{((s-1)/\log n)\log\delta} - 1/n = n^{-o(1)},$$

so that

$$\log(1/p(n)) = o(\log n). \tag{2}$$

Let $M = \lceil 2/p \rceil + 1$. Now by Lemma 9, for large n,

$$\mathbb{P}(G_{M^2 n',k''} \text{ is not connected}) \ge 1/2.$$

Furthermore,

$$c\log(M^2 n') < c\log(M^2 n) = c(1+o(1))\log n < \lfloor (c+\varepsilon/2)\log n \rfloor = k'',$$

for sufficiently large n, since $\log M = o(\log n)$ by (2). This contradicts the hypothesis. □

Our next aim is to investigate more closely the increase in k necessary to boost connectivity to s-connectivity. For simplicity consider the case $s = 2$. Since we know that we need $k = \Theta(\log n)$ for connectivity, Theorem 1 only shows that this increase is at most $\varepsilon \log n$. However, our main result is the following, which shows in particular that $6\sqrt{\log n}$ is sufficient.

Theorem 10. *Fix an increasing positive integer sequence $s = s(n) = o(\log n)$, with $s(n^2) \le 2s(n) - 1$. Let $k = k(n)$ be a function such that $G_{n,k}$ is connected* **whp**. *Then*

$$\limsup_{n\to\infty} \mathbb{P}(G_{n,k+\lfloor 6\sqrt{(s-1)\log n}\rfloor} \text{ is } s\text{-connected}) = 1.$$

Remark. The conditions on $s(n)$ hold in particular for constant $s \geq 1$, as well as any increasing $s = s(n)$ with $(s-1)/\log n$ decreasing monotonically to 0.

This is essentially our sharpest result, and shows that, for instance, if $k = k(n)$ is a function such that $G_{n,k}$ is connected **whp**, then for all $\varepsilon > 0$

$$\mathbb{P}(G_{n,k+\lfloor 6(\log n)^{3/4}\rfloor} \text{ is } \sqrt{\log n}\text{-connected}) > 1 - \varepsilon$$

for infinitely many values of n.

Let us attempt to prove Theorem 10 for $s = 2$, using Lemma 7. We start by assuming that $G_{n,k}$ is not 2-connected with probability 1/2, say, and apply the lemma to show that $G_{n(1-\delta),k-d+1}$ is not connected with probability at least $p = \delta(1-\delta)^2/2 - n(ek\delta/d)^d$. To obtain a contradiction we need p to be constant. With the tools developed so far, we can either use "sharpness in n" (Lemma 9) to increase p by increasing n (as in the proof Theorem 2), or we must take δ to be constant, which necessitates making d^d at least n, so that we need d to be at least about $\log n/\log\log n$. What we really need is "sharpness in k", so that we could increase p to a constant by decreasing k still further. We could then optimize the choices of δ and d. The trouble is that proving sharpness in k does not seem to be easy. However, the following lemma tells us that we "often" have sharpness in k.

Lemma 11. *Suppose that $K = K(n)$ and a decreasing function $p = p(n)$ are such that*

$$\mathbb{P}(G_{n,K(n)} \textit{ is not connected}) > p = \Omega(n^{-1/4}).$$

Then, if $K'(n) = K(n) - \lceil 4\log(4/p(n^2))\rceil + 1$ we have, for an increasing sequence of values of n,

$$\mathbb{P}(G_{n,K'(n)} \textit{ is not connected}) > 1/8. \tag{3}$$

Remark. For most applications, $\log(4/p(n^2))$ is within a constant factor of $\log(4/p(n))$.

Proof. We use the sharpness in n and an averaging argument. Suppose that (3) does not hold for any $n > n_0$, i.e.,

$$\mathbb{P}(G_{n,K'(n)} \text{ is not connected}) \leq 1/8, \tag{4}$$

for all $n > n_0$. Note that this implies that $p(n) < 1/8$ for all $n > n_0$.

We will choose $n_1 > n_0$ and $n_2 = n_1^2$ (the exact choice of n_1 will be given later). Let X be the set of pairs (n,k) with $n_1 \leq n \leq n_2$ such that

$$p(n) < \mathbb{P}(G_{n,k} \text{ is not connected}) \leq 1/8.$$

Before giving the formal proof we outline the main idea. For any n with $n_1 \leq n \leq n_2$ we are assuming that there are "many" values of k with the pair (n,k) contained in X. However, by sharpness in n, we know that for any fixed k there are only a "small" number of values of n such that the pair (n,k) is in X. Thus, by calculating the size of X in these two ways, we obtain a contradiction.

For technical reasons, we measure the size of X under a non-uniform weighting, rather than just using the cardinality of X. This is essentially due to the fact that $\log n$ and k are linearly related, so we aim to estimate the area X as represented in Figure 2. The proof is slightly more complicated than one would hope since we do not know that various functions are "well behaved". For instance, we do not know that $\mathbb{P}(G_{n,k}$ is not connected) is monotonic in n.

Now we return to the formal proof. First we define the weighted sum of the points of X: let

$$\begin{aligned} T &= \sum_{(n,k)\in X} \log(\tfrac{n+1}{n}) \\ &= \sum_{n_1 \leq n \leq n_2} \sum_k \log(\tfrac{n+1}{n})\ \mathbf{1}(p(n) < \mathbb{P}(G_{n,k} \text{ is not connected}) \leq 1/8). \end{aligned}$$

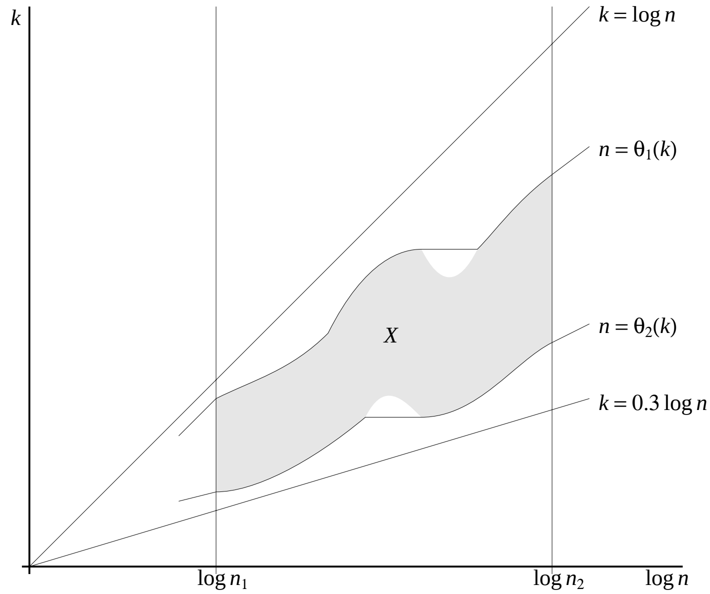

Figure 2: The region X and functions $\theta_1(k)$ and $\theta_2(k)$ in the proof of Lemma 10.

We will bound T in two different ways and obtain a contradiction.

First we bound T from below, using (4). Since (4) holds for any n with $n_1 \leq n \leq n_2$ there are at least $K(n) - K'(n) + 1 = \lceil 4\log(4/p(n^2)) \rceil$ values of k with $(n,k) \in X$. Thus

$$T \geq \sum_{n=n_1}^{n_2} \log(\tfrac{n+1}{n})\, 4\log(4/p(n^2)). \tag{5}$$

Next we bound T from above by using the sharpness in n. We split T up into many parts and bound each of these individually: let

$$\begin{aligned} T_k &= \sum_{n:(n,k)\in X} \log(\tfrac{n+1}{n}) \\ &= \sum_{n_1 \leq n \leq n_2} \log(\tfrac{n+1}{n})\ \mathbf{1}(p(n) < \mathbb{P}(G_{n,k} \text{ is not connected}) \leq 1/8), \end{aligned}$$

so that

$$T = \sum_k T_k. \tag{6}$$

(Note that, for all but finitely many k, T_k will be zero.) Let

$$\begin{aligned} \theta_1(k) &= \min\{n : \mathbb{P}(G_{n,k} \text{ is not connected}) > p(n)\,\} \\ \theta_2(k) &= \max\{n : \mathbb{P}(G_{n,k} \text{ is not connected}) \leq 1/8\,\}. \end{aligned}$$

Note that θ_1 and θ_2 exist since for any fixed k

$$\lim_{n\to\infty} \mathbb{P}(G_{n,k} \text{ is not connected}) = 1.$$

These are useful quantities since

$$X_k \subset [\theta_1(k), \theta_2(k)]$$

where $X_k = \{n : (n,k) \in X\}$. Also, both θ_1 and θ_2 are monotonically increasing in k, since $\mathbb{P}(G_{n,k}$ is not connected) is decreasing in k for fixed n.

Next we bound θ_1. By Lemma 6 we know that

$$\mathbb{P}(G_{n,\lceil \log n \rceil} \text{ is not connected}) = o(n^{-1/4}).$$

Since $p(n) = \Omega(n^{-1/4})$ we can choose M so that

$$\mathbb{P}(G_{n,\lceil \log n \rceil} \text{ is not connected}) \leq p(n) \tag{7}$$

for all $n \geq M$. This implies that, for all $n \geq e^M$,

$$\theta_1(\lceil \log n \rceil) > n, \tag{8}$$

since otherwise there exists an $n' \leq n$ and $k = \lceil \log n \rceil \geq M$ with

$$\mathbb{P}(G_{n',k} \text{ is not connected}) > p(n').$$

But clearly $n' > k \geq M$ and $k \geq \log n'$, contradicting (7) and monotonicity in k.

From the definition of $\theta_1(k)$ we have

$$\mathbb{P}(G_{\theta_1(k),k} \text{ is not connected}) > p(\theta_1(k)).$$

Applying Lemma 9 (sharpness in n) we have

$$\mathbb{P}(G_{N_0(k),k} \text{ is not connected}) > 1 - 1/e - o(1),$$

where $N_0(k) = (\lceil 2/p(\theta_1(k)) \rceil + 1)^2 \, \theta_1(k)$. Lemma 8 (monotonicity) then implies that, for any $N(k) \geq N_0(k)$,

$$\mathbb{P}(G_{N(k),k} \text{ is not connected}) > \tfrac{1}{4}(1 - 1/e) - o(1), \tag{9}$$

as long as $k > 0.3 \log N_0(k)$. If $k \leq 0.3 \log N_0(k)$, the last assertion follows from Theorem 3. Since $\frac{1}{4}(1 - 1/e) > 1/8$ there is a k_0 such that for all $k > k_0$ and $N(k) \geq N_0(k)$

$$\mathbb{P}(G_{N(k),k} \text{ is not connected}) > 1/8.$$

It follows that, for $k > k_0$,

$$\theta_2(k) + 1 \leq N_0(k) = \left(\left\lceil \tfrac{2}{p(\theta_1(k))} \right\rceil + 1 \right)^2 \theta_1(k) \leq \left(\tfrac{4}{p(\theta_1(k))} \right)^2 \theta_1(k). \tag{10}$$

Let $\hat{\theta}_1(k) = \max\{\theta_1(k), n_1\}$ and $\hat{\theta}_2(k) = \min\{\theta_2(k), n_2\}$. For any fixed k we have $\{n : (n,k) \in X\} \subset [\hat{\theta}_1(k), \hat{\theta}_2(k)]$ and thus that for any $k > k_0$

$$T_k \leq \sum_{\hat{\theta}_1(k)}^{\hat{\theta}_2(k)} \log(\tfrac{n+1}{n}) = \log(\hat{\theta}_2(k) + 1) - \log(\hat{\theta}_1(k)) \leq 2 \log(4/p(\hat{\theta}_1(k))). \tag{11}$$

Now we are ready to choose n_1. There exists an $n_3 = n_3(k_0)$ so that

$$\theta_2(k) < n_3 \text{ for all } k \leq k_0, \tag{12}$$

and thus that, as long as $n_1 \geq n_3$, $T_k = 0$ for all $k \leq k_0$, implying that (11) holds for all k. We choose $n_1 = \max\{n_0, n_3(k_0), e^M\}$. Also, if $\hat{\theta}_1(k) > n_2$ then $\hat{\theta}_1(k) > \hat{\theta}_2(k)$ and the above sum (11) is zero. Since $p(n)$ is decreasing, we have

$$T_k \leq 2\log(4/p(n_2))$$

for all k. Hence, $T_k = 0$ for all $k > \log n_2$ (since $\theta_1(\lceil \log n_2 \rceil) > n_2$ and θ_1 is increasing), and summing over k, we have

$$T \leq 2\log n_2 \log(4/p(n_2)). \tag{13}$$

To complete the proof we choose $n_2 = n_1^2$ and compare the two bounds (13) and (5). By (5)

$$T \geq \sum_{n=n_1}^{n_2} \log(\tfrac{n+1}{n})\ 4\log(4/p(n^2)) \geq 4\log(\tfrac{n_2+1}{n_1})\log(4/p(n_2)) > 2\log n_2 \log(4/p(n_2))$$

which contradicts Equation (13).

□

Theorem 10 is an immediate application of Lemma 11.

Proof of Theorem 10. The proof is similar to that of Theorem 1. Indeed our approach is exactly that described after the proof of Theorem 2, and it only remains to choose $\delta = \delta(n)$ and $d = d(n)$. Let $d = d(n) = \lceil 3\sqrt{(s-1)\log n}\rceil$, $\delta = \delta(n) = n^{-1.1/d} = e^{-1.1(\log n)/d}$, $k' = k + d - 1$ and $k'' = k' + d - 1$. As usual, we may assume $k'' \leq k + 6\sqrt{(s-1)\log n} < \log n$.

Suppose that $\limsup \mathbb{P}(G_{n,k''}$ is s-connected$) < 1$. Then there exists $\theta > 0$ and n_0 such that

$$\mathbb{P}(G_{n,k''} \text{ is not } s\text{-connected}) \geq \theta$$

for all $n \geq n_0$. By Lemma 7, the probability that $G_{n(1-\delta),k'}$ is not connected is at least

$$\theta\delta^{s-1}(1-\delta)^2 - n(ek''\delta/d)^d.$$

Now $d/\log n \to 0$, so $(ek''/d)^d \leq (d/e\log n)^{-d} = n^{-(d/\log n)\log(d/e\log n)} = n^{o(1)}$. Thus $n(ek''\delta/d)^d = n\delta^d n^{o(1)} = n^{-0.1+o(1)}$. Hence, by Lemma 8 (monotonicity) the probability that $G_{n,k'}$ is not connected is at least

$$\tfrac{\theta}{4}\delta^{s-1}(1-\delta)^2 - n^{-0.1+o(1)} = e^{-1.1(s-1)(\log n)/d + O(1)} - n^{-0.1+o(1)} = \omega(e^{-d/8}).$$

If we set

$$p(n) = 4e^{-d(n)/8},$$

then for sufficiently large n, $G_{n,k'}$ is not connected with probability at least $p(n)$. By assumption

$$d(n^2) = \lceil 3\sqrt{(s(n^2)-1)\log n^2}\rceil \leq \lceil 3\sqrt{2(s(n)-1)2\log n}\rceil \leq 2d(n),$$

so that

$$d(n) \geq d(n^2)/2 \geq 4\log(4/p(n^2)),$$

and, since $d(n)$ is an integer,

$$k = k' - d(n) + 1 \leq k' - \lceil 4\log(4/p(n^2))\rceil + 1.$$

Since $s(n)$ is increasing, $d(n)$ is increasing and $p(n)$ is decreasing in n. Applying Lemma 11 (sharpness in k) gives $\mathbb{P}(G_{n,k}$ is not connected$) > 1/8$ for infinitely many n, which is a contradiction. □

We expect that if $G_{n,k}$ is connected and s is constant, then one only needs to increase k by about $c(s-1)\log\log n$ to obtain s-connectivity. The following is a heuristic argument that supports this conjecture.

It seems likely (see [1]) that the obstructions to connectivity are small components, approximately circular in shape, containing around $k+1$ points, and surrounded by an annulus A of area about $C\log n$ containing no points, where C is some absolute constant. Call these *type 1* configurations. It also seems likely that the obstructions to s-connectivity are identical, except that A now contains $s-1$ points: call these *type s* configurations. A fixed type s configuration is $f(s) = (C\log n)^{s-1}/(s-1)!$ times as likely to occur as its corresponding type 1 configuration, so that if we expect approximately one type 1 configuration in $S_{nf(s)}$, we also expect around $f(s)$ type s configurations in $S_{nf(s)}$ and hence one type s configuration in S_n. This suggests that $G_{n,k}$ becomes s-connected at about the same k that makes $G_{nf(s),k}$ connected. Suppose the critical k for connectivity is given approximately by $c\log n$. One would then expect that the k needed to make $G_{nf(s),k}$ connected is about $c\log(nf(s)) - c\log n = c\log f(s)$ larger than the k needed to make $G_{n,k}$ connected. Thus if $G_{n,k}$ is connected, one would expect that increasing k by about $c\log f(s) \sim c(s-1)\log\log n$ would give s-connectivity.

4 s-Coverage

Let $\mathcal{P}_n$ be a Poisson process of intensity one in the square S_n. For any $x \in \mathcal{P}_n$, let $r(x,k)$ be the distance from x to its kth nearest neighbour (infinite if this does not exist), and let $B_k(x) = \{y \in S_n : d(y,x) \le r(x,k)\}$. We say that $\mathcal{P}_n$ is a (k,s)-cover if each point of S_n lies in at least s of the regions $B_k(x)$.

The following theorem is the analogue of Theorem 10 from [1], and has an essentially identical proof. The graph $\vec{G}_{n,k}$ is defined exactly as $G_{n,k}$, except that we place *directed* edges pointing away from each point towards its k nearest neighbours. For a directed graph $\vec{G}$, $\delta_{\text{in}}(\vec{G})$ denotes the minimum in-degree of $\vec{G}$.

Theorem 12. *Suppose that* $k = \lfloor c\log n\rfloor$ *is such that* **whp** $\delta_{\text{in}}(\vec{G}_{n,k}) \ge s = s(n)$. *Then, for any* $\varepsilon > 0$*, letting* $k' = \lfloor (c+\varepsilon)\log n\rfloor$ *we have that* **whp** $\mathcal{P}_n$ *is a* (k', s)*-cover. Conversely, suppose that* **whp** $\mathcal{P}_n$ *is a* (k,s)*-cover for* $k = \lfloor c\log n\rfloor$*. Then, for any* $\varepsilon > 0$*, letting* $k' = \lfloor (c+\varepsilon)\log n\rfloor$ *we have that* **whp** $\delta_{\text{in}}(\vec{G}_{n,k'}) \ge s$.

This result will enable us to deduce results about s-coverage from the corresponding results on the minimum in-degree. We can prove exact analogues of the s-connectivity results for the minimum in-degree, which will be enough to deduce a version of Theorem 2 for s-coverage.

The following is immediate from the proof of Theorem 3 of [1].

Theorem 13. *If* $c \le 0.7209$ *then* $\mathbb{P}(\delta_{\text{in}}(\vec{G}_{n,\lfloor c\log n\rfloor}) = 0) \to 1$*, and hence* $\mathbb{P}(\vec{G}_{n,\lfloor c\log n\rfloor}$ *is connected*$) \to 0$*, as* $n \to \infty$*. If* $c \ge 0.9967$ *then* $\mathbb{P}(\vec{G}_{n,\lfloor c\log n\rfloor}$ *is connected*$) \to 1$*, and hence*

$$\lim_{n\to\infty} \mathbb{P}(\delta_{\text{in}}(\vec{G}_{n,\lfloor c\log n\rfloor}) = 0) = 0.$$

We first show that, as long as $s = o(\log n)$, $\delta_{\text{in}}(\vec{G}_{n,k}) \ge s$ occurs "just after" $\delta_{\text{in}}(\vec{G}_{n,k}) \ge 1$ as k increases. First, we establish the result for constant s. To do this, we need a lemma which is exactly analogous to Lemma 7.

Lemma 14. *For any* $s, d, k, n \in \mathbb{N}$ *and* $0 < \delta < 1$ *with* $0 < d \le k$ *and*

$$\mathbb{P}(\delta_{\text{in}}(\vec{G}_{n,k}) < s) \ge \theta,$$

we have

$$\mathbb{P}(\delta_{\text{in}}(\vec{G}_{n(1-\delta),k-d+1}) = 0) \geq \theta\delta^{s-1}(1-\delta) - n(ek\delta/d)^d.$$

Proof. If $\delta_{\text{in}}(\vec{G}_{n,k}) < s$ then we have a set S of (at most) $s-1$ elements in $\mathcal{P}_n$ whose removal creates a vertex v of in-degree zero in $G_{n,k}$. We follow the proof and notation of Lemma 7, noting that $v \in \mathcal{P}_{(1-\delta)n}$ and $S \subset \mathcal{P}_{\delta n}$ occurs with probability at least $\delta^{s-1}(1-\delta)$. Thus

$$\mathbb{P}(\delta_{\text{in}}(\vec{G}'') = 0 \mid \delta_{\text{in}}(\vec{G}) < s) \geq \delta^{s-1}(1-\delta).$$

Consequently,

$$\mathbb{P}(\delta_{\text{in}}(\vec{G}'') = 0) \geq \theta\delta^{s-1}(1-\delta).$$

The proof that $\vec{G}'$ is a subgraph of $\vec{G}''$ with probability at least $1 - n(ek\delta/d)^d$ is exactly as in the proof of Lemma 7. As in that proof, we obtain

$$\mathbb{P}(\delta_{\text{in}}(\vec{G}') = 0) \geq \theta\delta^{s-1}(1-\delta) - n(ek\delta/d)^d.$$

□

The next lemma is analogous to Lemma 8.

Lemma 15. *Suppose that n and $k \geq 0.7\log n$ are such that $\mathbb{P}(\delta_{\text{in}}(\vec{G}_{n,k}) = 0) > p$ for some $p = p(n)$. Then, for any $n' > n$ and any K,*

$$\mathbb{P}(\delta_{\text{in}}(\vec{G}_{n',k}) = 0) > p/4 - O(n^{-K}).$$

Proof. Fix n'. Consider the square $S \subset S_{n'}$ of area n in the bottom left hand corner of $S_{n'}$. Let $\vec{G}$ be the directed k nearest neighbour graph formed by the points in S. The induced subgraph $\vec{H}$ of $\vec{G}_{n',k}$ formed by the vertices in S is a subgraph of $\vec{G}$. By hypothesis, with probability at least p, $\delta_{\text{in}}(\vec{G}) = 0$. Hence, with probability at least $p/4$, $\vec{G}$ contains a vertex v of in-degree zero in the bottom left hand quarter of S (by symmetry).

Divide the square S into 25 small squares. By Theorem 13 we may assume $k \leq \log n$ and so, as in the proof of Lemma 8, with probability $1 - O(n^{-K})$, the top 10 and right 10 squares (a total of 16 squares) each contain at least k points. In this case there will be no directed edge from $S_{n'} \setminus S$ to any point in the bottom left 9 squares and thus v is a vertex of in-degree zero in the original graph. □

Next we have the promised result for minimum in-degree s, for constant s.

Theorem 16. *Fix $s \in \mathbb{N}$. Suppose $k = k(n)$ is such that* **whp** $\delta_{\text{in}}(\vec{G}_{n,k}) \geq 1$. *Then, for any $\varepsilon > 0$,* **whp** $\delta_{\text{in}}(\vec{G}_{n,\lfloor k(1+\varepsilon)\rfloor}) \geq s$.

Proof. As for Theorem 1, using Lemmas 14 and 15 and the bounds in Theorem 13 in place of Lemmas 7 and 8 and the bounds in Theorem 3. □

We now extend Theorem 16 to the case $s = o(\log n)$. First we need a lemma, which is analogous to Lemma 5.

Lemma 17. *Assume $c' > 0$ is independent of n and $k = k(n) \geq 0.7\log n$. The probability that there exists a vertex of in-degree zero in $\vec{G}_{n,k}$ within distance $c'\sqrt{\log n}$ of two sides of S_n is $o(n^{-1/20})$.*

Proof. Suppose that we have a vertex v of in-degree zero within distance $c'\sqrt{\log n}$ of two sides of S_n. Let w be the closest point of $V(\vec{G}_{n,k}) \setminus \{v\}$ to v and write $\rho = d(v,w)$ for the distance between them. One of the right angled isosceles triangles with hypotenuse vw lies inside S_n: call it T. T has area $\rho^2/4$ and can contain no vertices of $\vec{G}$. On the other hand, there are at least k points in $A = \{x \in S_n : d(x,w) \leq \rho,\ d(x,v) \geq \rho\}$, since otherwise w would send an edge to v. Therefore, there must be at least k points in $A \cup T$, which must all lie in $A \setminus T$. The probability of this happening is at most

$$\left(\frac{|A \setminus T|}{|A \cup T|}\right)^k \leq \left(\frac{|A|}{|A| + |T|}\right)^k \leq \left(\frac{\pi\rho^2}{\pi\rho^2 + \rho^2/4}\right)^k \leq \left(1 + \frac{1}{4\pi}\right)^{-k}.$$

The number of choices for v is $O(\log n)$ and, given v, there are $O(\log n)$ choices for w, making $o((\log n)^2)$ choices for both, so that the probability that we have such a configuration is at most $O((\log n)^2 (1 + \frac{1}{4\pi})^{-k}) \leq o(n^{-1/20})$, since $k \geq 0.7 \log n$. □

We will need the following lemma in the proof of our sharpness result.

Lemma 18. *For any $k > 1.1 \log n$ the probability that $\delta_{\text{in}}(\vec{G}_{n,k}) = 0$ is $o(n^{-1/20})$.*

Proof. We refer to the proof of Theorem 8 in [1]. Set $\gamma = (\frac{4\pi}{3} + \frac{\sqrt{3}}{2})(\frac{\pi}{3} + \frac{\sqrt{3}}{2})^{-1}$ and $\gamma' = (\frac{5\pi}{6} + \frac{\sqrt{3}}{2})(\frac{\pi}{3} + \frac{\sqrt{3}}{2})^{-1}$. We see that the probability of a vertex of in-degree zero near to no side of S_n is $n^{1+o(1)}\gamma^{-k} = o(n^{-1/20})$, and that the probability of a small component near to exactly one side of S_n is $n^{1/2+o(1)}\gamma'^{-k} = o(n^{-1/20})$. Combining this with Lemma 17 the result follows. □

Now we prove the analogue of Lemma 9.

Lemma 19. *Suppose that $k = k(n)$ and $p = p(n)$ are such that*

$$\mathbb{P}(\delta_{\text{in}}(\vec{G}_{n,k}) = 0) > p = \Omega(n^{-1/20}).$$

Then

$$\mathbb{P}(\delta_{\text{in}}(\vec{G}_{n',k}) = 0) > 1 - 1/e - o(1)$$

where $n' = (\lceil 2/p \rceil + 1)^2 n$.

Proof. First note that if $k < 0.7209 \log n$ then $k < 0.7209 \log n'$ and, thus, by Theorem 3 $\delta_{\text{in}}(\vec{G}_{n',k}) = 0$ **whp**, and the lemma is trivially true. Thus we can assume $k \geq 0.7209 \log n$.

As before, we say that a point $x \in V(\vec{G}_{n,k})$ is *close* to a side s of S_n if x is less than distance $c'\sqrt{\log n}$ from s, where $c' = c'(0.7, 1)$ is as in Lemma 4. We call x *central* if it is not close to any side s of S_n. We know that with probability at least p, $\vec{G}_{n,k}$ contains a vertex v of in-degree zero, which can be close to at most two sides of S_n. Write α for the probability that v is central, β for the probability that v is close to exactly one side of S_n, and γ for the probability that v is close to two sides of S_n (so that it lies at a corner of S_n). We have $\alpha + \beta + \gamma \geq p$, and, by Lemma 17, we may assume that either $\alpha > \frac{p}{8}$ or $\beta > \frac{p}{2}$. If we specify one side s of S_n, the probability that v is either central or only close to s is thus at least $\frac{p}{8}$.

Let $M = \lceil 2/p \rceil + 1$. We consider the larger square $S_{M^2 n}$, and tessellate it with copies of S_n. We only consider the small squares of the tessellation incident with the boundary of $S_{M^2 n}$. Considering sides of these copies of S_n lying on the boundary of $S_{M^2 n}$, we see that we have $4(M-1)$ independent opportunities to obtain a vertex v of in-degree zero in one of the small squares S, in such a way that v can only be close to the boundary of S if it is close to the boundary of $S_{M^2 n}$. Such a vertex will also have in-degree zero in $\vec{G}_{M^2 n,k}$, since, by Lemma 4, **whp** no edge of $\vec{G}_{M^2 n,k}$ has length greater than $c'\sqrt{\log n'}$. Therefore, if p' is the probability that $\delta_{\text{in}}(\vec{G}_{M^2 n,k}) = 0$, we have

$$p' \geq 1 - \left(1 - \tfrac{p}{8}\right)^{4(M-1)} - o(1) > 1 - e^{-\frac{p}{2}(M-1)} - o(1) \geq 1 - 1/e - o(1).$$

□

Theorem 20. *Let $s = s(n) = o(\log n)$. Suppose c is such that* **whp** $\delta_{\text{in}}(\vec{G}_{n,\lfloor c\log n\rfloor}) \geq 1$. *Then, for any $\varepsilon > 0$,* **whp** $\delta_{\text{in}}(\vec{G}_{n,\lfloor (c+\varepsilon)\log n\rfloor}) \geq s$.

Proof. As for Theorem 2, using Lemmas 14 and 19 in place of Lemmas 7 and 9. □

We may now deduce the following result on s-coverage.

Theorem 21. *Let $s = s(n) = o(\log n)$. Suppose $c > 0$ is such that* **whp** *$\mathcal{P}_n$ is a $(\lfloor c\log n\rfloor, 1)$-cover. Then, for any $\varepsilon > 0$,* **whp** *$\mathcal{P}_n$ is a $(\lfloor (c+\varepsilon)\log n\rfloor, s)$-cover.*

Proof. Apply Theorem 12 and Theorem 20. □

The proofs of the following results are almost identical to those of their counterparts for connectivity, so we omit them.

Lemma 22. *Suppose that $k = k(n)$ and a decreasing function $p = p(n)$ are such that*

$$\mathbb{P}(\delta_{\text{in}}(\vec{G}_{n,k}) = 0) > p = \Omega(n^{-1/20}).$$

Then, setting $k' = k - \lceil 4\log(4/p(n^2))\rceil + 1$ we have, for infinitely many n,

$$\mathbb{P}(\delta_{\text{in}}(\vec{G}_{n,k'}) = 0) > 1/8.$$

□

Theorem 23. *Fix a non-decreasing positive sequence $s = s(n) = o(\log n)$, with $s(n^2) \leq 2s(n) - 1$. Let $k = k(n)$ be a function such that $\delta_{\text{in}}(\vec{G}_{n,k(n)}) \geq 1$* **whp**. *Then*

$$\limsup_{n\to\infty} \mathbb{P}\left(\delta_{\text{in}}(\vec{G}_{n,k+\lfloor 6\sqrt{(s-1)\log n}\rfloor}) \geq s\right) = 1.$$

□

5 Open problems

Many open problems remain in this area. The one most relevant to this paper is to improve the bound in Theorem 10. Specifically, suppose that for some $k(n)$ we know that $G_{n,k(n)}$ is connected **whp**. We would like to know the "smallest" function $f(n,s)$ such that $G_{n,k(n)+f(n,s)}$ is s-connected **whp**. As we mentioned following the proof of Theorem 10, we suspect that $f(n,s) = c(s-1)\log\log n$ is enough, where c is the critical constant for the k-nearest neighbour model from [2]. More precisely, we make the following conjecture.

Conjecture 1. *Let c be the critical constant for the k-nearest neighbour model, and let $c' > c$. Is it true, for any $s \in \mathbb{N}$ and $k(n)$ such that $G_{n,k(n)}$ is connected* **whp**, *that $G_{n,k(n)+\lfloor c'(s-1)\log\log n\rfloor}$ is s-connected* **whp**?

Perhaps a sharper version of Lemma 11 might help in this direction. Also open is the determination of the critical constant c for both connectivity and coverage.

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