

Graph Theory: Projects

October 11, 2008

I chose these projects because I think they are all interesting: they are of different levels of difficulty, and I will take this into account when grading your presentations. Choose a project you like. (I will find some way of dealing with conflicts, should they arise.) The aim is to have 2 people to each project: you are both required to read and understand the proof(s), and either one or both of you can present it in class.

Vizing's theorem

Vizing's theorem states that for any graph G ,

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$$

where $\Delta(G)$ is the maximum degree of G and $\chi'(G)$ is the edge chromatic number of G . This is stated (for regular graphs) on page 32 of Hartsfield and Ringel. There is a proof on pages 153–154 of *Modern Graph Theory* by Bollobás.

Graphs with high girth and high chromatic number

Intuitively, it seems that if a graph has high chromatic number, it must have lots of short cycles. However, it is possible (but very hard – try it) to construct a graph with arbitrarily high girth and arbitrarily high chromatic number. In 1959, Paul Erdős proved the existence of such graphs *without constructing any*, using what is known as the *probabilistic method* (which he had invented). At that time, no constructions of such graphs were known (and the problem had received a lot of attention).

My favourite exposition of this proof is in *The Probabilistic Method* by Alon and Spencer. It appears in all three editions: in the third edition it is on pages 41–42. Although

the proof is short, you should spend a lot of time making sure that you really understand what is going on.

Google

One of the ideas behind Google was to model the web as a graph, and to imagine a “random surfer” clicking on links (i.e. traversing edges) at random. Imagine that the surfer spends only a billionth of a second on each page, and surfs for, say, a week. Then the proportion of time spent by the random surfer on a particular webpage (vertex) is related to its PageRank, which ranks webpages in order of “importance”. This determines the order in which the search results appear on the screen.

This project is different from the others in that it is more open-ended. Also, it has more of a linear algebra flavour. A good place to begin would be with the papers below.

[1] K. Bryan and T. Leise, The \$25,000,000,000 eigenvector: the linear algebra behind Google, <http://www.rose-hulman.edu/~bryan/googleFinalVersionFixed.pdf> (accessed 10 October 2008).

[2] S. Brin and L. Page, The anatomy of a large-scale hypertextual web search engine, <http://www-db.stanford.edu/~backrub/google.html> (accessed October 10, 2008).

Stable marriages

Hall’s marriage theorem gives a necessary and sufficient condition for the existence of matchings in bipartite graphs. It has many applications in combinatorics, algebra and analysis. One way of formulating it is in terms of arranging marriages between m girls and $n \geq m$ boys. What if we go one step further and insist that these marriages are *stable*? In this setup each girl (resp. boy) separately ranks each of the boys (resp. girls) in order of preference, and we aim to arrange the marriages so that if girl x is *not* married to boy y , then either x is already married to someone she prefers to y , or y is already married to someone he prefers to x . Such a stable system of marriages can in fact be arranged. All this probably sounds incredibly silly, but it is in fact very important and has many applications. One such application is to *list colouring*.

All the material for this project is in *Modern Graph Theory* by Bollobás – the basic theorem is on pages 85–91, and the application to list colouring (also part of the project) is on pages 161–165. The two papers below might also be of interest.

[1] D. Gale and L.S. Shapley, College admissions and the stability of marriage, *American Mathematical Monthly* **69** (1962), 9–15.

- [2] T. Slivnik, A short proof of Galvin's theorem on the list-chromatic index of a bipartite multigraph, *Combinatorics, Probability and Computing* **5** (1996), 91–94.
-

Crossing numbers

The *crossing number* $\text{cr}(G)$ of a graph G is the smallest number of crossings in a drawing of G in the plane, so that, for instance, $\text{cr}(G) = 0$ iff G is planar. The special case $\text{cr}(K_{m,n})$ is known as *Turán's brick factory problem* and dates from 1944, when Paul Turán was working in a brick factory in a forced-labour camp during the Second World War. A more recent application is to VLSI design in computer science. Crossing numbers are the subject of Section 9.1 of Hartsfield and Ringel.

If G has n vertices and m edges, and if $m > 4n$, then $\text{cr}(G) \geq m^3/64n^2$. This is one version of the *crossing number inequality*. In 1995, László Székely discovered that this inequality yields very short proofs of some previously very hard results. For example, he used it to prove that, given n points in the plane, one of them determines at least $cn^{4/5}$ distinct distances from the others. Both the basic inequality and the application are part of the project.

- [1] L.A. Székely, Crossing numbers and hard Erdős problems in discrete geometry, *Combinatorics, Probability and Computing* **6** (1997), 353–358.
-

Koebe's theorem

This beautiful theorem is stated and illustrated on page 173 of Hartsfield and Ringel. It states that every planar graph is a *coin graph*. In other words, given a planar graph $G = (V, E)$, we can represent each $v \in V$ by a circle C_v in the plane so that if $uv \in E$ then C_u and C_v touch (are tangent to each other), and if $uv \notin E$ then C_u and C_v are disjoint. I have not seen Koebe's original proof (it is written in German), but there is a proof on pages 96–99 of *Combinatorial Geometry* by Pach and Agarwal. The proof contains some statements left as exercises, which you will have to do.

You might be interested in reading the biography of Koebe at

<http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Koebe.html>

(but this is not part of the project).
