Graph Theory: Projects

October 6, 2017

These projects are of different levels of difficulty, and I will take this into account when grading your presentations, so choose a project you like. The aim is to have 2 or 3 people to each project; you are each required to read and understand the proof(s), and each of you should present something in class.

Squaring the square

Can a square be dissected into a finite number of smaller squares, each with *different* side-lengths? The answer turns out to be yes, and the first examples were found by Sprague in 1939, and by Brooks, Smith, Stone and Tutte in 1940 (when all four were undergraduates). There's a surprising connection between this problem and electrical network theory, which is described in Chapter 2 of *Modern Graph Theory* by Bollobás. For the project, you should read the first two sections of that chapter, and also the two survey articles listed below.

Tutte later became a very influential graph theorist; his *1-factor theorem* is taught in many introductory courses (but not this one), and the *Tutte polynomial* is useful not just in graph theory, but also in knot theory and statistical mechanics. Tutte's work in cryptography during the Second World War enabled the British to read high-level German army messages; this work, which remained secret until very recently, has been described as one of the greatest intellectual feats of the war. There's an excellent documentary about this entitled "Bletchley Park's Lost Heroes" - just Google the title and the Youtube link will appear. (Update (2016): Youtube has now removed the movie, so the heroes are lost once again.)

[1] W.T. Tutte, The quest of the perfect squared square, *American Mathematical Monthly* **72** (1965), 29–35.

[2] N.D. Kazarinoff and R. Weitzenkamp, Squaring rectangles and squares, *American Mathematical Monthly* **80** (1973), 877–888.

Google

One of the ideas behind Google was to model the web as a graph, and to imagine a "random surfer" clicking on links (i.e. traversing edges) at random. Imagine that the surfer spends only a billionth of a second on each page, and surfs for, say, a week. Then the proportion of time spent by the random surfer on a particular webpage (vertex) is related to its PageRank, which ranks webpages in order of "importance". This determines the order in which the search results appear on the screen.

This project is different from the others in that it is more open-ended. Also, it has more of a linear algebra flavour. A good place to begin would be with the papers below.

[1] K. Bryan and T. Leise, The $25,000,000,000 eigenvector: the linear algebra behind Google, `http://www.rose-hulman.edu/~bryan/googleFinalVersionFixed.pdf` (accessed 6 October 2017).

[2] S. Brin and L. Page, The anatomy of a large-scale hypertextual web search engine, `http://ilpubs.stanford.edu:8090/361/1/1998-8.pdf` (accessed 6 October 2017).

Stable marriages

Hall's marriage theorem gives a necessary and sufficient condition for the existence of matchings in bipartite graphs. It has many applications in combinatorics, algebra and analysis. One way of formulating it is in terms of arranging marriages between m girls and $n \geq m$ boys. What if we go one step further and insist that these marriages are *stable*? In this setup each girl (resp. boy) separately ranks each of the boys (resp. girls) in order of preference, and we aim to arrange the marriages so that if girl x is *not* married to boy y, then either x is already married to someone she prefers to y, or y is already married to someone he prefers to x. Such a stable system of marriages can in fact be arranged, using the *Gale-Shapley algorithm*. All this probably sounds incredibly silly, but it's used, for instance, in matching organ donors to patients. In fact, Shapley and A.E. Roth were awarded the 2012 Nobel Prize in Economics; Shapley for the Gale-Shapley algorithm itself (Gale died in 2008), and Roth for applying the algorithm to economic markets.

The theory is presented in *Modern Graph Theory* on pages 85–91, but you must also read the classic original paper below.

[1] D. Gale and L.S. Shapley, College admissions and the stability of marriage, *American Mathematical Monthly* **69** (1962), 9–15.

Crossing numbers

The *crossing number* $\mathrm{cr}(G)$ of a graph G is the smallest number of crossings in a drawing of G in the plane, so that, for instance, $\mathrm{cr}(G) = 0$ iff G is planar. The special case $\mathrm{cr}(K_{m,n})$ is known as *Turán's brick factory problem* and dates from 1944, when Paul Turán was working in a brick factory in a forced-labour camp during the Second World War. A more recent application is to VLSI design in computer science.

If G has n vertices and m edges, and if $m > 4n$, then $\mathrm{cr}(G) \geq m^3/64n^2$. This is one version of the *crossing number inequality.* In 1995, László Székely discovered that this inequality yields very short proofs of some previously very hard results. For example, he used it to prove that, given n points in the plane, one of them determines at least $cn^{4/5}$ distinct distances from the others. Both the basic inequality and the application are part of the project.

[1] L.A. Székely, Crossing numbers and hard Erdős problems in discrete geometry, *Combinatorics, Probability and Computing* **6** (1997), 353–358.

Koebe's theorem

This beautiful theorem states that every planar graph is a *coin graph.* In other words, given a planar graph $G = (V, E)$, we can represent each $v \in V$ by a circle C_v in the plane so that if $uw \in E$ then C_u and C_w touch (are tangent to each other), and if $uw \notin E$ then C_u and C_w are disjoint. I have not seen Koebe's original proof (it is written in German), but there is a proof on pages 96–99 of *Combinatorial Geometry* by Pach and Agarwal. The proof contains some statements left as exercises, which you will have to do.

You might be interested in reading the biography of Koebe at

`http://www-history.mcs.st-andrews.ac.uk/Biographies/Koebe.html`

(but this is not part of the project).

[1] J. Pach and P.K. Agarwal, *Combinatorial Geometry*, Wiley, 1995.