

Geometry I

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Unlike in the high school competitions (e.g. USA Math Olympiad), traditional Euclidean geometry does not seem to loom large in the Putnam. Instead, you are likely to encounter questions like this:

1. (Putnam 1988) a) If every point of the plane is painted one of three colors, do there necessarily exist two points of the same color exactly one inch apart?
b) What if “three” is replaced by “nine”?

I want to use this and the next question to illustrate a very basic problem-solving principle:

- **Try to think of a simpler version of the problem, which captures the main features of the original problem.**

Let’s return to the original question. First of all, I expect nobody has asked you a question like this before. Where do you begin? Well, let’s begin by replacing “three” by “two”. If every point of the plane is painted one of two colors, do there necessarily exist two points of the same color exactly one inch apart? Perhaps we could approach this by considering just one point, which we can suppose is colored red. If the conclusion is false, what does that tell us? And then what? Notice that we are starting in the most pedestrian way imaginable, just to get some feel for the problem.

2. (Putnam 2002) Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

What is a simpler version of this? How about four points on a circle? What can we say about them?

Sometimes, problems that appear to be about geometry are actually about something else. Here is an example, to see if you learned anything last week!

3. (Putnam 1971) Let there be given nine lattice points (points with integral coordinates) in three dimensional Euclidean space. Show that there is a lattice point on the interior of one of the line segments joining two of these points.

And here is another example.

4. (Putnam 1993) Show there do not exist four points in the Euclidean plane such that the pairwise distances between the points are all odd integers.

While this is certainly a geometry problem, it looks to me as if algebra and even number theory are likely to be helpful in solving it.

Homework

1. (Not a Putnam problem, but one of my favorites.) Any four points in the plane determine six possible distances between pairs of points. List all the configurations of four points which determine only two *distinct* distances. An example is a unit square: there are four distances of 1, and two of $\sqrt{2}$ (the diagonals).

2. (Putnam 1994) Show that if the points of an isosceles right triangle of (small) side length 1 are each colored with one of four colors, then there must be two points of the same color which are at least a distance $2 - \sqrt{2}$ apart.

[**Hint.** You can do this problem by considering only seven points, three of which are the vertices of the triangle.]