Difference equations

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The basic method

A typical linear difference equation is

$$a_{n+2} = 5a_{n+1} - 6a_n \tag{1}$$

In order for (1) to define a sequence, we need to know the values of n for which it is true (suppose it is true for $n \geq 1$), and also two initial values (for a three-term equation such as this). Suppose, for instance, that $a_1 = a_2 = 1$.

The idea is to look for solutions of the form $a_n = \lambda^n$. Substituting this into (1) and dividing through by λ^n yields the *characteristic equation*

$$\lambda^2 = 5\lambda - 6 \tag{2}$$

with solutions $\lambda = 2$ and $\lambda = 3$. Therefore, the general solution to (1) is

$$a_n = A2^n + B3^n$$

and you can use the fact that $a_1 = a_2 = 1$ to solve for A and B.

Exercise: find a formula for the Fibonacci numbers.

Repeated roots

Sometimes the characteristic equation has repeated roots. This would happen, for example, if we had started with the difference equation

$$a_{n+2} = 6a_{n+1} - 9a_n \tag{3}$$

whose characteristic equation is

$$\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0$$

This time the general solution is

$$a_n = A3^n + Bn3^n$$

You should prove for yourself that this works by substituting it into (3). As before, you have to solve for A and B using whatever initial conditions you are given.

An important example of this case arises in the analysis of *gambler's ruin*. You start with \$37, and throw a fair coin repeatedly: if it's heads you win \$1, if it's tails you lose \$1. When you are down to \$0 you are *ruined*, but if you ever get to \$100 you stop playing and declare yourself a winner. What is the probability that you are eventually ruined? Why?

You should be able to guess and prove a formula for the general solution when the characteristic equation is something like

$$(\lambda - 2)^3(\lambda - 3)(\lambda - 5)^2$$

Inhomogeneous equations

A typical example is

$$a_{n+2} = 5a_{n+1} - 6a_n + n^2 \qquad (4)$$

The presence of the n^2 term is what makes the equation "inhomogeneous". Now, ignoring any boundary conditions for the moment,

• Any solution of the corresponding homogeneous equation (in this case (1)) can be added to any solution of the original inhomogeneous equation (i.e. (4)) to get a new solution of the inhomogeneous equation

(Why ?) This is exactly analogous to the corresponding situation in linear algebra ($\mathbf{Ax} = \mathbf{b}$ vs. $\mathbf{Ax} = \mathbf{0}$) and in differential equations (of which more later). And, as in those situations, we now have to find a *particular solution*. For polynomial terms (n^2 in the above example), the basic rule is to guess a polynomial

$$a_n = An^2 + Bn + C \qquad (5)$$

of the same degree as the inhomogeneous term (in this case 2), substitute (5) into (4), compare coefficients of n^2, n and 1, and solve for A, B and C. This gives $A = 1/2, B = 3/2$ and $C = 5/2$, so that

$$a_n = \frac{n^2 + 3n + 5}{2}$$

is a solution of (4). Therefore, the general solution of (4) is

$$a_n = \frac{n^2 + 3n + 5}{2} + D2^n + E3^n$$

and now you can use the initial values (or boundary conditions) to find D and E, exactly as before.

Sometimes this doesn't work: then you have to increase the degree of your guess. A simple example is

$$a_{n+1} = a_n + n \tag{6}$$

with, say, $a_1 = 0$. Guessing $a_n = An + B$ doesn't work (try it). So you have to try $a_n = An^2 + Bn + C$, which works (try that too). Of course, the a_n are just the triangular numbers. If we had

$$a_{n+2} = 2a_{n+1} - a_n + n \tag{7}$$

then neither linear nor quadratic guesses work, so we have to try a cubic, which works.

The reason we run into problems is quite interesting. Equations (6) and (7) can be rewritten as

$$a_{n+1} - a_n = n \tag{8}$$

and

$$(a_{n+2} - a_{n+1}) - (a_{n+1} - a_n) = n \tag{9}$$

respectively, which in turn are a bit like the *differential equations*

$$y' = x \tag{10}$$

and

$$y'' = x \tag{11}$$

respectively. This "explains" (i.e. you should explain, based on this) why we need to go up to quadratics and cubics. In fact, every difference equation we have looked at has a differential equation analogue: I'll leave the details to you.

Homework

1. (Putnam 1980) For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion

$$u_{n+1} = 2u_n - n^2$$

have $u_n > 0$ for all $n \geq 0$? (Express the answer in the simplest form.)

2. (Putnam 1950) The sequence $x_0, x_1, x_2, \ldots$ is defined by the conditions $x_0 = a, x_1 = b$,

$$x_{n+1} = \frac{x_{n-1} + (2n-1)x_n}{2n} \quad \text{for } n \geq 1,$$

where a and b are given numbers. Express $\lim_{n\to\infty} x_n$ concisely in terms of a and b.