Inequalities

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These are *sometimes* needed in the Putnam. Here is a basic toolkit.

- **Sum of squares inequality**

$$\frac{1}{n}\sum_{i=1}^{n} a_i \leq \left(\frac{1}{n}\sum_{i=1}^{n} a_i^2\right)^{1/2}$$

- **Arithmetic-Mean–Geometric-Mean (AM/GM) Inequality** For $a_i > 0$,

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{1}{n}\sum_{i=1}^{n} a_i$$

- **Cauchy-Schwarz Inequality**

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}$$

If you know Linear Algebra, then the Cauchy-Schwarz Inequality simply amounts to the fact that $\mathbf{a} \cdot \mathbf{b} \leq ||\mathbf{a}||||\mathbf{b}||$ for two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$. Alternatively, you can prove it by expanding both sides, but you will then need to use the $n = 2$ case of the AM/GM Inequality (which you should also be able to prove). One way to prove the full AM/GM Inequality is to use **Jensen's Inequality** (see later). The first inequality above, probably the most useful, is just a special case of the Cauchy-Schwarz Inequality.

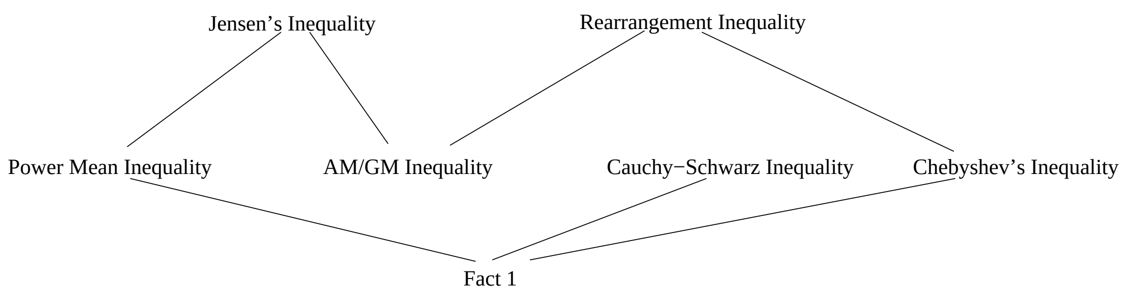

Figure 1: Some famous inequalities

Sometimes none of these do the trick and you have to use something stronger, for instance the

• **Power Mean Inequality** For $a_i > 0$ and $r < s$ with $r \neq 0$ and $s \neq 0$

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i^r\right)^{1/r} \leq \left(\frac{1}{n}\sum_{i=1}^{n} a_i^s\right)^{1/s}$$

In some sense (exercise) the AM/GM Inequality is the special case $r = 0, s = 1$, even though, as stated, this case doesn't seem to make any sense. The sum of squares inequality is of course just the case $r = 1, s = 2$. All three inequalities are consequences of

• **Jensen's Inequality** If f is convex ("concave-up") on an interval I and $a_i \in I$ then for weights λ_i summing to 1

$$f\left(\sum_{i=1}^{n} \lambda_i a_i\right) \leq \sum_{i=1}^{n} \lambda_i f(a_i)$$

The case $n = 2$ is the *definition* of convexity, and the general case is not hard to prove by induction (exercise). It is interesting that such a powerful inequality has such a short proof: the big idea was to realize that such a statement might be true and would, if true, be useful. The case $0 < r \leq s$ of the Power Mean Inequality and the full AM/GM Inequality follow from the convexity, for $x > 0$, of the functions $f(x) = x^{s/r}$ and $f(x) = -\log x$ respectively.

The following inequality is frequently useful.

• **Rearrangement Inequality** If $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ then for all permutations σ of $\{1, 2 \ldots, n\}$

$$\sum_{i=1}^{n} a_{\sigma_i} b_i \leq \sum_{i=1}^{n} a_i b_i$$

You should prove this for $n = 2$: the proof of the general case uses the same idea. An easy consequence is

• **Chebyshev's Inequality** If $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ then

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right) \leq \frac{1}{n}\sum_{i=1}^{n} a_i b_i$$

It is also possible, but a bit tricky, to prove the AM/GM Inequality using the Rearrangement Inequality. (The trick is to consider the numbers

$$a_1/G, a_1 a_2/G^2, \ldots, a_1 a_2 \cdots a_n/G^n = 1,$$

where G is the geometric mean.)

Other useful inequalities are the **Triangle Inequality** $||\mathbf{a}+\mathbf{b}|| \leq ||\mathbf{a}||+||\mathbf{b}||$ for vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, which you can prove for yourself, **Hölder's Inequality** and **Minkowski's Inequality**. But we'll discuss these some other time.

Examples

1. Prove that if $a > 0$ then

$$a + \frac{1}{a} \geq 2.$$

2. (Nesbitt's Inequality) Prove that, for $a, b, c > 0$,

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}.$$

3. (Putnam 1975) Show that if $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, then
   (a) $n(n+1)^{1/n} < n + s_n$ for $n > 1$, and
   (b) $(n-1)n^{-1/(n-1)} < n - s_n$ for $n > 2$.

4. (Putnam 1977) Suppose that $a_1, a_2, \ldots, a_n$ are real ($n > 1$) and

$$A + \sum_{i=1}^{n} a_i^2 < \frac{1}{n-1}\left(\sum_{i=1}^{n} a_i\right)^2.$$

Prove that $A < 2a_i a_j$ for $1 \leq i < j \leq n$.

Homework

1. (Putnam 1961) Let $\alpha_1, \alpha_2, \alpha_3, \ldots$ be a sequence of positive real numbers; define s_n as $(\alpha_1 + \alpha_2 + \cdots + \alpha_n)/n$ and r_n as $(\alpha_1^{-1} + \alpha_2^{-1} + \cdots + \alpha_n^{-1})/n$. Given that $\lim s_n$ and $\lim r_n$ exist as $n \to \infty$, prove that the product of these limits is not less than 1.

2. (Putnam 1988) Prove or disprove: if x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.