The pigeonhole principle

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- **Basic version** $n+1$ items, n boxes $\Rightarrow$ some box contains at least 2 items
- **Advanced version** $mn+1$ items, n boxes $\Rightarrow$ some box contains at least $m+1$ items

Neither of these is particularly profound, but the trick is to know how to use them.

Examples

1. (Dirichlet, 1842, original application) Let α be a real number, and let $\varepsilon > 0$. Then there exist integers p and $q > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{\varepsilon}{q}.$$

Proof. Let $M > \frac{1}{\varepsilon}$. Apply the (basic) pigeonhole principle with items $0, \{\alpha\}, \{2\alpha\}, \ldots, \{M\alpha\}$ and boxes $[0, 1/M), [1/M, 2/M), \ldots, [(M-1)/M, 1]$.

2. (Putnam 1990) Prove that any convex pentagon whose vertices (no three of which are collinear) have integer coordinates must have area $\geq 5/2$.

3. (Putnam 1993) Let $x_1, x_2, \ldots, x_{19}$ be positive integers each of which is less than or equal to 93. Let $y_1, y_2, \ldots, y_{93}$ be positive integers each of which is less than or equal to 19. Prove that there exists a (nonempty) sum of some x_i's equal to a sum of some y_j's.

4. (Putnam 1994) Let A and B be 2×2 matrices with integer entries such that $A, A+B, A+2B, A+3B$, and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.

Homework

1. (Putnam 2000) Let a_j, b_j, c_j be integers for $1 \leq j \leq N$. Assume, for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of $j, 1 \leq j \leq N$.

2. (Putnam 2006) Prove that, for every set $X = \{x_1, x_2, \ldots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$