Graph Theory

November 16, 2012

Class Problems

1. (Putnam 1953) Six points are in general position in space (no three in a line, no four in a plane). The fifteen line segments joining them in pairs are drawn and then painted, some segments red, some blue. Prove that some triangle has all its sides the same color.

A **graph** G is a set $V = V(G)$ of **vertices**, together with a subset $E = E(G)$ of the set of unordered pairs of elements of V. The members of E are called **edges**. Graphs can be finite or infinite, but let's just consider finite graphs for now, i.e., let's suppose $|V|$ is finite (in which case so is $|E|$). Vertices x and y are said to be **adjacent** if $\{x, y\} \in E$, so that there is an edge e "joining" them. The edge e is usually written xy rather than $\{x, y\}$, and is said to be **incident** with its **endvertices** x and y. The **degree** $d(x)$ of a **vertex** x (this is the singular of "vertices") is the number of edges incident with x. We normally visualize graphs by drawing them: below is a graph with 10 vertices and 11 edges.

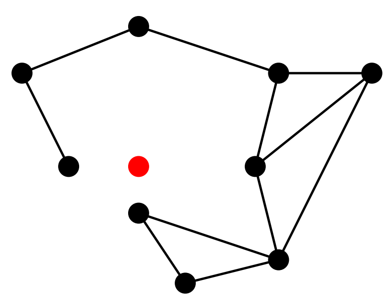

Figure 1: A graph

Here's an easy exercise: prove that, in any finite graph, the sum of the degrees of the vertices (22 in the graph above) is always twice the number of edges. By the way, the **order** and **size** of a graph G are the numbers of vertices and edges of G, respectively. A graph of order n can have at most $\binom{n}{2}$ edges: if it has all $\binom{n}{2}$ edges, it is said to be **complete** and written K_n. (K stands for one of *Komplett, Kazimierz or Kuratowski* - I don't know which.) A graph is **connected** if there's a **path** between any two of its vertices (a path is what you think it is); the maximal connected **subgraphs** (a subgraph is also what you think it is) of a graph are called its **components**. The graph above has 2 components, one of which is an **isolated vertex**. A graph with no **cycles** (guess what these are) is said to be **acyclic**, or a **forest**, and a connected forest is called a **tree**.

So, to get back to the Putnam question we started with, you have to show that if the edges of the complete graph K_6 are each colored either red or blue, then the colored K_6 contains either a red triangle, or a blue triangle (or both). If you can do this, you will have proved a special case of **Ramsey's theorem**, which states that, for all k, there exists an $N = N(k)$, such that if the edges of K_N are colored either red or blue, then the colored K_N contains either a red K_k (with all $\binom{k}{2}$ edges red), or a blue K_k. This theorem is one of the cornerstones of **Ramsey Theory** – another fundamental result in Ramsey Theory is the **Hales-Jewett theorem**, which I mention because one of its discoverers, R.I. Jewett, retired from teaching here at WWU only 2 years ago, and indeed he used to teach this very class. His coauthor, A.W. Hales, was one of the winners (top 5 scorers) of the Putnam in 1958 and 1959.

But I digress. Here's another problem from the early days of the Putnam.

2. (Putnam 1956) Consider a set of $2n$ points in space, $n > 1$. Suppose they are joined by at least $n^2 + 1$ segments. Show that at least one triangle is formed. Show that for each n it is possible to have $2n$ points joined by n^2 segments without any triangles being formed.

In other words, prove that a graph of order $2n$ and size at least $n^2 + 1$ contains a triangle, K_3. This result, **Mantel's theorem**, is a special case of **Turán's theorem**, which gives the maximum number $f(n, r)$ of edges in a graph of order n without a K_r. Turán's theorem marks the start of what is known as **Extremal Graph Theory**.

For a hint on Mantel's theorem, suppose that xy is an edge in a graph G of order $2n$ and size m with no triangles. Prove that $d(x) + d(y) \leq 2n$. Now prove that

$$\sum_{x \in V(G)} d(x)^2 = \sum_{xy \in E(G)} (d(x) + d(y)) \leq 2nm.$$

Finally, apply the Cauchy-Schwarz inequality, together with the result of the easy exercise at the top of the page. This is one of the many proofs of the theorem.

The most useful proof technique in graph theory is **mathematical induction**. Both Ramsey's theorem and Turán's theorem are best proved using induction. Another useful technique is **double counting**, which means counting the same thing in two different ways. But that's another story, which can wait.

Homework

1. (Putnam 1958) Let a complete oriented graph on n points be given, i.e., a set of n points $1, 2, 3 \ldots, n$, and between any two points i and j a direction, $i \to j$. Show that there exists a permutation of the points, $[a_1, a_2, a_3, \ldots, a_n]$, such that $a_1 \to a_2 \to a_3 \to \cdots \to a_n$.

2. (Putnam 1965) At a party, assume that no boy dances with every girl but each girl dances with at least one boy. Prove that there are two couples gb and $g'b'$ which dance whereas b does not dance with g' nor does g dance with b'.

3. (Putnam 1968) Prove that a list can be made of all the subsets of a finite set in such a way that i) the empty set is first in the list, ii) each subset occurs exactly once, iii) each subset in the list is obtained either by adding one element to the preceding subset or by deleting one element of the preceding subset.

4. (Putnam 1990) Let G be a finite group of order n generated by a and b. Prove or disprove: there is a sequence $g_1, g_2, g_3, \ldots, g_{2n}$ such that every element of G occurs exactly twice, and g_{i+1} equals $g_i a$ or $g_i b$, for $i = 1, 2, \ldots, 2n$. (Interpret g_{2n+1} as g_1.)

Fun problems to think about over Thanksgiving

5. (Putnam 1958) Given a set of $n + 1$ positive integers, none of which exceeds $2n$, show that at least one member of the set must divide another member of the set.

6. (Putnam 1962) Given five points in a plane, no three of which lie on a straight line, show that some four of these points form the vertices of a convex quadrilateral.

7. (Putnam 1971) Let c be a real number such that n^c is an integer for every positive integer n. Show that c is a non-negative integer.