Coins, cards and dice

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There are two sides to probability: probabilistic intuition and technical skill. It's a good idea to develop both. Having some idea of the approximate answer to a question (using intuition) quite often narrows down the list of techniques that might be useful in solving it. Of course this isn't always true, otherwise mathematics would be a lot easier than it is. Let's start with one of the first questions on probability to appear on the Putnam exam.

1. (Putnam 1960) A player throwing a die scores as many points as on the top face of the die and is to play until his score reaches or passes a total n. Denote by $p(n)$ the probability of making exactly the total n, and find the value of $\lim_{n\to\infty} p(n)$.

It's not clear from the wording whether you have to prove that your answer is correct, so let's just ignore that part. But you should be able to guess what the limit is. Try thinking, for instance, about what the score is after m throws of the die. There will be about $\frac{m}{6}$ ones, $\frac{m}{6}$ twos, and so on. So the score will be about $\frac{m}{6}(1+2+3+4+5+6)$. Of course this is a different problem. But how does it relate to the original one?

Here's a problem from an earlier exam which doesn't mention the word "probability", but which certainly involves probability.

2. (Putnam 1958) What is the average straight line distance between two points on a sphere of radius 1?

I just have a comment about this one - rather than considering two points, is there a way to consider just one? (Of course, this might not be the best way to solve the problem. For instance, how would I find the average *square* of the same distance?)

Now for a harder question. Putnam questions in probability seem to get harder as we move closer to the present.

3. (Putnam 1974, modified) An unbiased coin is tossed n times. What is the expected value of $|H - T|$, where H is the number of heads and T is the number of tails?

Calculating the expected value of $(H - T)^2$ is fairly straightforward (why?), but unfortunately this doesn't help us very much (why not?). However, at least it gives us some idea of how large $|H - T|$ should be. In fact, knowledge of this, and of the approximate size of the middle binomial coefficients, and of the expected values for $n \leq 8$ (say), and some luck... might just be enough to guess the answer, which can then be verified by induction. Well, maybe I'm stretching the truth a bit.

Here are some even harder questions, with hints for each.

4. (Putnam 1989) If α is an irrational number, $0 < \alpha < 1$, is there a finite game with an honest coin such that the probability of one player winning the game is α? (An honest coin is one for which the probability of heads and the probability of tails are both $\frac{1}{2}$. A game is finite if with probability 1 it must end in a finite number of moves.)

Hint 1. The answer is yes.
Hint 2. How can I convert an infinite sequence of heads and tails into a real number between 0 and 1?

5. (Putnam 1993) Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players, A and B. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n+1$. The last person to discard wins the game. Assuming optimal strategy by both A and B, what is the probability that A wins?

Hint 1. Solve the cases $n = 2$ and $n = 3$, and make a guess.
Hint 2. Your guess is correct. Why?

Homework

These all relate to Question 2 above. Let S be a sphere of radius 1.

- What is the expected value of the square of the straight line distance between two points on S?
- What is the expected value of the distance "as the plane flies" between two points on S?
- What are the answers to all these questions in d dimensions?