Counting

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Recall that the number of ways of choosing r objects from n (without replacement, where the order does not matter) is denoted $\binom{n}{r}$ and pronounced "n choose r". These *binomial coefficients* have many properties, for instance

- $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ for $1 \le r \le n-1$
- $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$
- $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
- $p | \binom{p}{r}$ if p is prime and $1 \le r \le p-1$

As an example of some of these facts in action, we can prove that

- $\sum_{r=0}^{s} \binom{m}{r}\binom{n}{s-r} = \binom{m+n}{s}$

The **Fibonacci numbers** are defined by

- $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 3$

You should prove, for instance, that the number of ways of tiling a $1 \times n$ rectangle with squares and "dominoes" is F_{n+1}. There is a connection between Fibonacci numbers and binomial coefficients, namely,

- $F_n = \sum_r \binom{n-r}{r-1}$

which can be useful and which you should prove by induction.

Examples

1. (Putnam 1990) If X is a finite set, let $|X|$ denote the number of elements in X. Call an ordered pair (S,T) of subsets of $\{1,2,\ldots,n\}$ *admissible* if $s > |T|$ for each $s \in S$, and $t > |S|$ for each $t \in T$. How many admissible ordered pairs of subsets of $\{1,2,\ldots,10\}$ are there? Prove your answer.

2. (Putnam 1991) Suppose p is an odd prime. Prove that

$$\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

3. (Putnam 1992) For nonnegative integers n and k, define $Q(n,k)$ to be the coefficient of x^k in the expansion of $(1+x+x^2+x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-2j}.$$

4. (Putnam 1996) Define a *selfish* set to be a set which has its own cardinality (number of elements) as a subset. Find, with proof, the number of subsets of $\{1,2,\ldots,n\}$ which are minimal selfish sets, that is, selfish sets none of whose proper subsets is selfish.

5. (Putnam 2000) Prove that the expression

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

Homework

1. (Putnam 1985) Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

$$A_1 \cup A_2 \cup A_3 = \{1,2,3,4,5,6,7,8,9,10\},$$

and

$$A_1 \cap A_2 \cap A_3 = \emptyset,$$

where $\emptyset$ denotes the empty set. Express the answer in the form $2^a 3^b 5^c 7^d$, where a, b, c, and d are nonnegative integers. [**Hint.** Draw a Venn diagram.]

2. (Putnam 1987) Let r, s and t be integers with $0 \le r, 0 \le s$, and $r+s \le t$. Prove that

$$\frac{\binom{s}{0}}{\binom{t}{r}} + \frac{\binom{s}{1}}{\binom{t}{r+1}} + \cdots \frac{\binom{s}{s}}{\binom{t}{r+s}} = \frac{t+1}{(t+1-s)\binom{t-s}{r}}.$$

[**Hint.** Use induction on s.]