Draw a diagram

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The following 6 problems all involve probability. They also have another thing in common.

1. (Putnam 1989) A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $(a\sqrt{b} - c)/d$, where a, b, c, d are positive integers.

2. (Putnam 1961) Let α and β be given positive real numbers, with $\alpha < \beta$. If two points are selected at random from a straight line segment of length β, what is the probability that the distance between them is at least α?

3. (Putnam 1970) Three numbers are chosen independently at random, one from each of the three intervals $[0, L_i]$ $(i = 1, 2, 3)$. If the distribution of each random number is uniform with respect to length in the interval it is chosen from, determine the expected value of the smallest of the three numbers chosen.

4. (Putnam 1968) The temperatures in Chicago and Detroit are x and y respectively. These temperatures are not assumed to be independent; namely, we are given i) $\mathbb{P}(x = 70)$, ii) $\mathbb{P}(y = 70)$, and iii) $\mathbb{P}(\max(x, y) = 70)$. Determine $\mathbb{P}(\min(x, y) = 70)$.

5. (Putnam 1993) Two real numbers x and y are chosen at random in the interval $(0, 1)$ with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers.

6. (Putnam 1982) Let p_n be the probability that $c + d$ is a perfect square when the integers c and d are selected independently at random from the set $\{1, 2, \ldots, n\}$. Show that $\lim_{n\to\infty}(p_n\sqrt{n})$ exists and express this limit in the form $r(\sqrt{s} - t)$, where s and t are integers and r is a rational number.

The common thread is that, for each of these problems, one should draw some type of diagram. In fact, these diagrams even look fairly similar - as you will discover if you draw them yourself. The diagrams do get progressively more complicated though!

Some hints

It is very natural to draw a diagram of the target in Problem 1, but Problem 2 can also be attacked geometrically, by representing the choice of points x and y as a single point with coordinates (x, y). For Problem 3, the analogous diagram is three-dimensional, but why not solve the two-dimensional version first?