Mathematical induction: variants and subtleties

October 29, 2010

Mathematical induction is one of the most useful techniques for solving problems in mathematics. I'm assuming you're familiar with the basic method and its modifications (e.g. "strong induction"), and also that you know why induction doesn't prove that all horses are the same color.

The impression one gets from introductory courses is that induction is a machine for automatically proving anything, and my intention is to convince you that there's a lot more to it than that. Induction doesn't always work, and even when it does, considerable skill might be required to get it to do the job.

Let's start with a simple example, that was set in a mathematical competition for 8th graders in Leningrad (in 1988).

- Suppose that n positive numbers $x_1, x_2, \ldots, x_n$ satisfy

$$x_1 + x_2 + \cdots + x_n = \frac{1}{2}.$$

Prove that

$$\frac{(1-x_1)}{(1+x_1)} \frac{(1-x_2)}{(1+x_2)} \cdots \frac{(1-x_n)}{(1+x_n)} \geq \frac{1}{3}.$$

The main idea here is just the idea of using induction at all (rather than some general inequality, for instance). But, even when you've had this idea, you have to apply it in the right way. Simply writing down the inequality with $n = k$ and multiplying both sides by $(1 - x_{k+1})/(1 + x_{k+1})$ will not do, for several reasons. For a start, if $x_1 + x_2 + \cdots + x_k = \frac{1}{2}$ and $x_{k+1} > 0$, then $x_1 + x_2 + \cdots + x_k + x_{k+1} > \frac{1}{2}$. Also, $\frac{1}{3}(1 - x_{k+1})/(1 + x_{k+1})$ will actually be less than $\frac{1}{3}$, so the induction step fails.

What follows is a list of variants and subtleties of mathematical induction. Before we begin, let me note two things. First, while most textbooks present the inductive step as "going from $n = k$ to $n = k + 1$", many working mathematicians see it as a process of *reduction*, i.e., of reducing the case $n = k$ to the case $n = k - 1$. This might seem like pure semantics, but most of what follows will make more sense if you adopt the more advanced viewpoint, and sometimes the advanced viewpoint is essential. Second, I'm not going to bother setting $n = k$ in problems just involving n, because I assume that you basically know what you're doing.

1   Back and forth induction

One of the simplest instances of this arises in the proof of the AM/GM inequality we saw last week. Rather than repeat the entire proof, I'll just talk about its structure. Writing $P(n)$ for the statement of the inequality with n positive numbers, the proof consists of (I) dismissing $P(1)$ as trivial; (II) establishing $P(2)$; (III) showing that, for $n \geq 2$, $P(n)$ implies $P(2n)$ and (IV) deducing $P(n-1)$ from $P(n)$, where again $n \geq 2$. Note that step (III) could in principle be replaced by showing that (III′) $P(n)$ is true for infinitely many values of n. The point is that the truth of, say, $P(5)$ can be traced back to the truth of $P(2)$ via (III) (or (III′)) and (IV). This is different from the usual induction recipe, but, then again, induction isn't a recipe, it's an *ingredient*.

2   Nested induction

Suppose we have to show that, for $n \geq 1$, $2 \cdot 7^n + 3 \cdot 5^n - 5$ is divisible by 24. Naturally, we try induction. There are various ways to do the algebra, but in one way, for the induction step to work, we need the fact that $5 + 7^n$ is divisible by 6 for $n \geq 1$. This in turn is best proved by a separate induction. So we have an induction within an induction, which of course is perfectly OK – the "inner" induction makes no reference to the "outer" one.

3   Double induction

All the statements we've had to prove so far have involved just one positive integer variable n. If we have a statement involving two or more positive integer variables, we might proceed in various ways. To illustrate this, consider the following problem:

- Let f be a function of two positive integer variables with $f(1, 1) = 2$ and

$$f(m + 1, n) = f(m, n) + 2(m + n)$$
$$f(m, n + 1) = f(m, n) + 2(m + n - 1)$$

for all $m, n \geq 1$. Prove that

$$f(m, n) = (m + n)^2 - (m + n) - 2n + 2 \tag{1}$$

for all $m, n \geq 1$.

We don't have to invent a new method – we just have to apply the old method twice. For example, we could set $n = 1$ and prove, by induction on m, that $f(m, 1) = (m + 1)^2 - (m + 1) + 2$. Then, with m fixed, we treat (1) as a statement involving just one variable n, and, having just established the base case, we proceed by induction on n. Try it.

3.1 Choice of induction variable

Here's a statement involving two positive integer variables:

$$\frac{(m + n)!}{m! n!} \quad \text{is an integer}$$

We know that the left hand side is just the binomial coefficient $\binom{m+n}{n}$, but let's just forget this and try to prove the statement by induction. Somehow, the above strategy doesn't quite work (try it). The trick is to do induction on $m + n$: we aim to deduce an example of the statement with, say, $m + n = 100$ from one or more examples of the statement with $m + n = 99$. And this actually turns out to be quite easy (in part because we have just rediscovered the "combinatorial" proof). Now: prove that the product of any r consecutive positive integers is divisible by $r!$.

In a problem involving many positive integer variables, simply selecting the appropriate combination of variables to "induct on" is an art in itself.

4 Strengthening the induction hypothesis

Let's try proving that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2$$

using induction on n. Obviously this doesn't work (otherwise it would not be in this section). But if we instead try proving that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n},$$

again using induction on n, this time it works. What's going on? Why is it *easier* to prove a *stronger* statement?

The answer lies in the fact that, in the inductive step, while the conclusion has got stronger, *so has the induction hypothesis*. We have more to prove, but more to prove it with. Somehow, the stronger statement is better matched to the inductive step.

Fine, but how would we know that we should strengthen the induction hypothesis in this way, and not in some other way? In the example above, perhaps it isn't obvious. So let's discuss another example. With F_n denoting the nth Fibonacci number (so that $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$), suppose we have to prove that

$$F_n^2 + F_{n+1}^2 = F_{2n+1}$$

for $n \geq 1$. Let's call that statement $P(n)$. Assuming it, and trying to prove $P(n + 1)$, we find we need to show that

$$2F_n F_{n+1} + F_{n+1}^2 = F_{2n+2},$$

again for $n \geq 1$. Let's call *that* statement $Q(n)$. Trying to prove $Q(n)$ by induction, we find we need to know that

$$F_{n+1}^2 + F_{n+2}^2 = F_{2n+3}$$

But this is just $P(n + 1)$. Our inner induction loop has broken out and somehow become entangled with the outer induction. Is this a disaster? No! We just have to *strengthen the original statement* to

$$F_n^2 + F_{n+1}^2 = F_{2n+1} \quad \textbf{and} \quad 2F_n F_{n+1} + F_{n+1}^2 = F_{2n+2}$$

Since $P(n)$ and $Q(n)$ together imply $P(n + 1)$, and $Q(n)$ and $P(n + 1)$ together imply $Q(n + 1)$, if $R(n)$ is the statement that both $P(n)$ and $Q(n)$ are true, then $R(n)$ implies $R(n + 1)$. So we are done, once we know that $R(1)$ holds (and it does).

A lot of theorems get proved this way. Two other examples are the proof of Fermat's Last Theorem for $n = 4$, and the proof that splitting fields are unique up to isomorphism.

Homework

1. (Putnam 1953) Show that the sequence

$$\sqrt{7}, \sqrt{7 - \sqrt{7}}, \sqrt{7 - \sqrt{7 + \sqrt{7}}}, \sqrt{7 - \sqrt{7 + \sqrt{7 - \sqrt{7}}}}, \ldots$$

converges, and evaluate the limit.

2. (Putnam 1966) Justify the statement that

$$3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + \cdots}}}}}$$

[**Note and hint.** This problem, submitted by Ramanujan, appeared in the Journal of the Indian Mathematical Society around 1913. And now for the hint: $3 = \sqrt{1 + 2\sqrt{1 + 3 \cdot 5}}$.]

3. (Putnam 1967) Show that the sum of the first n terms in the binomial expansion of $(2 - 1)^{-n}$ is $\frac{1}{2}$, where n is a positive integer.

4. (Putnam 1997) Players $1, 2, 3, \ldots, n$ are seated round a table, and each has a single penny. Player 1 passes a penny to Player 2, who then passes two pennies to Player 3. Player 3 then passes one penny to Player 4, who passes two pennies to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.