Linear algebra done wrong

October 8, 2010

Determinants are currently unfashionable for a number of reasons – they are viewed as computational rather than conceptual, they require an exponential amount of time to compute (and to typeset), the list goes on. Nonetheless, they come up quite frequently in the Putnam (e.g. on last year's exam), so it's good to be familiar with their properties, some of which I've listed below. (I'm assuming you're aware of the "standard" theory, including the definition via cofactor expansions, $\det I = 1, \det A^{\mathsf{T}} = \det A, \det AB = \det A \det B$ and so on.) Think of all this as a sophisticated version of Sudoku. To begin with:

- The determinant, viewed as a function of any of its columns, is linear.
- If any two columns (or rows) of a determinant are equal, the determinant is zero.

The first fact implies that the determinant is also linear when viewed as a function of any of its rows (since transposing a matrix converts rows into columns and preserves the determinant). (Indeed, for all the facts below, "column" can be replaced by "row", and the statement is still true.) We continue with some consequences:

- Adding a multiple of one column to another does not change the determinant.
- A determinant changes sign if we interchange two columns.

These few facts are usually all we need. For example, we can use them, in conjunction with the cofactor expansion, to solve the following two problems.

1. (Putnam 1969) Let D_n be the determinant of order n of which the element in the i^{th} row and the j^{th} column is the absolute value of the difference of i and j. Show that D_n is equal to

$$(-1)^{n-1}(n-1)2^{n-2}.$$

2. (Putnam 1978) Let $a, b, p_1, p_2, \ldots, p_n$ be real numbers with $a \neq b$. Define $f(x) = (p_1 - x)(p_2 - x)(p_3 - x) \cdots (p_n - x)$. Show that

$$\det \begin{bmatrix} p_1 & a & a & a & \cdots & a & a \\ b & p_2 & a & a & \cdots & a & a \\ b & b & p_3 & a & \cdots & a & a \\ b & b & b & p_4 & \cdots & a & a \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b & b & b & b & \cdots & p_{n-1} & a \\ b & b & b & b & \cdots & b & p_n \end{bmatrix} = \frac{bf(a) - af(b)}{b - a}.$$

Homework

1. (Putnam 1992) Let D_n denote the value of the $(n-1) \times (n-1)$ determinant

$$\det \begin{bmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{bmatrix}.$$

Is the set $\{D_n/n!\}_{n \geq 2}$ bounded?

2. (Putnam 1995) To each positive integer with n^2 decimal digits, we associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for $n = 2$, to the integer 8617 we associate

$$\det \begin{bmatrix} 8 & 6 \\ 1 & 7 \end{bmatrix} = 50.$$

Find, as a function of n, the sum of all the determinants associated with n^2-digit integers. (Leading digits are assumed to be nonzero; for example, for $n = 2$, there are 9000 determinants.) [**Hint.** Use linearity.]