High-dimensional geometry

October 1, 2010

Let's start with the following question:

What's the volume of a 10-dimensional ball of radius 1?

Obviously, the first step is to work out what a 10-dimensional ball of radius 1 actually is. Then, we have to figure out what the volume of such a thing is supposed to mean.

Well, what's a 2-dimensional ball of radius 1? It's a disc of radius 1, which is the set of points (x, y) in $\mathbb{R}^2$ satisfying $x^2 + y^2 \leq 1$. And a 3-dimensional ball of radius 1 is just the set of points (x, y, z) in $\mathbb{R}^3$ satisfying $x^2 + y^2 + z^2 \leq 1$. So, a 10-dimensional ball of radius 1 should be the set of points $(x_1, \ldots, x_{10})$ in $\mathbb{R}^{10}$ satisfying $x_1^2 + \cdots x_{10}^2 \leq 1$.

So, how do we define the volume of a 10-dimensional ball? The same way we define the volume of a 3-dimensional ball: we cut it up into slices, perpendicular to one of the coordinate axes, say, and add up the volume of each slice. In 3 dimensions, if we use slices perpendicular to the z-axis, then each slice has volume

$$\pi(\sqrt{1 - z^2})^2 \, dz$$

so the total volume is just

$$\int_{-1}^{1} \pi(\sqrt{1 - z^2})^2 \, dz = \frac{4}{3}\pi$$

as you know. The $\sqrt{1 - z^2}$ term is just the radius of the cross-sectional disc at distance z above (or below) the xy-plane. In a similar way, the volume of each 9-dimensional slice of a 10-dimensional unit-radius ball should be

$$V_9(\sqrt{1 - z^2})^9 \, dz$$

where V_9 is the volume of a 9-dimensional ball of unit radius. (Why?) And so, if V_n denotes the volume of an n-dimensional ball of unit radius, we have

$$V_{10} = V_9 \int_{-1}^{1} (\sqrt{1 - z^2})^9 \, dz$$

At this point, the problem is basically solved, except we still have to evaluate the integral. But you should be able to do this, and hence (by recursion) solve the original problem.

I don't want to spoil the fun of working out a closed-form expression for the answer (things actually work out quite nicely), but I will tell you that its numerical value is about 2.55, which seems (to me) somewhat small. Even more surprising is that the volume of a ball of unit radius in 100 dimensions is about 2.4×10^{-40}. What's going on?

In 2 dimensions, it's easy to see that the area of the disc $x^2 + y^2 \leq 1$ should be at least half the area of the square "surrounding it", and it seems that the same should be true in 3 dimensions, which it is, but not for the same reason: the octahedron $|x| + |y| + |z| \leq 1$ sitting inside the unit ball $x^2 + y^2 + z^2 \leq 1$ actually has volume $\frac{4}{3}$, not 4. In higher dimensions, the ball still contains the "octahedron", but the volume of the "octahedron" gets really small with increasing dimension. **Exercise:** Compare the volumes of a cube of side length 2, a ball of radius 1, and the (generalized) octahedron with vertices at $(\pm 1, 0, \ldots, 0), (0, \pm 1, 0, \ldots, 0), \ldots, (0, 0 \ldots, 0, \pm 1)$ in $\mathbb{R}^n$.

In any case, it seems that a high-dimensional ball of radius 1 occupies a very small proportion of its surrounding cube. One way of seeing why this should be is to think about n-dimensional volume in a different way. We can pick n numbers $X_1, \ldots, X_n$ at random between -1 and 1. Then, we can calculate the quantity

$$Q_n = X_1^2 + X_2^2 + \cdots + X_n^2$$

and see whether or not this is less than 1. If we do this, say, a million times (easy on a computer), then the proportion of times that Q_n is less than 1, multiplied by the volume 2^n of the surrounding ("circumscribing") cube, will approximate the volume of the ball. (Why?) Now, if X_i is a uniform random variable on $[-1, 1]$, then $\mathbb{E}(X_i^2) = \frac{1}{3}$ and $\mathrm{Var}(X_i^2) = \frac{4}{45}$, so that $\mathbb{E}(Q_n) = \frac{n}{3}$ and $\mathrm{Var}(Q_n) = \frac{4n}{45}$. Consequently, by the Central Limit Theorem, it's rather unlikely that Q_n is less than 1. This at least explains why the volume of a unit-radius ball is much much smaller than 2^n. It doesn't actually explain why the volume tends to zero, so **you can think about this**.

High-dimensional geometry is pretty counterintuitive. For example, it's possible to fit an elephant into a cube *of side length 1 inch* (in sufficiently high dimension). But it isn't possible to put the same elephant into a ball of unit radius, in any dimension.

Homework

I couldn't actually find any questions about high-dimensional geometry on past Putnam exams. However, on reading many Putnam questions, you might have the feeling that you really need to know more about some mathematical topic or other, in order to even *understand* the question, let alone solve it – just as for the question we started with. But in fact, for many such questions, you probably already know everything you need to know.

The questions below are all taken from the 1955 exam. Choose two, and think about each for at least an hour, or until you solve it.

1. Prove that there is no set of integers m, n, p except $0, 0, 0$ for which $m + n\sqrt{2} + p\sqrt{3} = 0$.

2. On a circle, n points are selected and the chords joining them in pairs are drawn. Assuming that no three of these chords are concurrent (except at the endpoints), how many points of intersection are there?

3. A sphere rolls along two intersecting straight lines. Find the locus of its center.

4. Do there exist 1,000,000 consecutive integers each of which contains a repeated prime factor?

5. Given an infinite sequence of 0's and 1's and a fixed integer k, suppose that there are no more than k distinct blocks of k consecutive terms. Show that the sequence is eventually periodic. (For example, the sequence 11011010101, followed by alternating 0's and 1's indefinitely, which is periodic beginning with the fifth term.)