Symmetric polynomials

December 2, 2009

The following pre-war problem will serve to illustrate the theory. (Actually the problem goes back to Isaac Newton, who solved it in 1665.)

1. (Putnam 1939) Find the cubic equation whose roots are the cubes of the roots of

$$x^3 + ax^2 + bx + c = 0.$$

Let's call the roots p, q and r. Then

$$(x-p)(x-q)(x-r) = x^3 + ax^2 + bx + c$$

so that

$$\begin{aligned} p + q + r &= -a \\ pq + qr + rp &= b \\ pqr &= -c \end{aligned}$$

Now the cubic we want is

$$(x-p^3)(x-q^3)(x-r^3) = x^3 - (p^3+q^3+r^3)x^2 + (p^3q^3+q^3r^3+r^3p^3)x - p^3q^3r^3$$

so our task is to express the *symmetric polynomials* $p^3+q^3+r^3, p^3q^3+q^3r^3+r^3p^3$ and $p^3q^3r^3$ in terms of the *elementary symmetric polynomials* $s_1 = p+q+r, s_2 = pq+qr+rp$ and $s_3 = pqr$. Of course, $p^3q^3r^3 = (pqr)^3 = (-c)^3 = -c^3$, but how about the other two?

Let's ask a simpler version of the question: can we express $p^2 + q^2 + r^2$ and $p^2q^2 + q^2r^2 + r^2p^2$ in terms of the elementary symmetric polynomials? The basic method here is *experimentation*. First of all, both these expressions involve squares, so we compute

$$(p + q + r)^2 = p^2 + q^2 + r^2 + 2pq + 2qr + 2rp$$

from which we see that

$$p^2 + q^2 + r^2 = (p + q + r)^2 - 2(pq + qr + rp) = s_1^2 - 2s_2 = a^2 - 2b$$

and

$$(pq + qr + rp)^2 = p^2q^2 + q^2r^2 + r^2p^2 + 2pqr(p + q + r)$$

so that

$$p^2q^2 + q^2r^2 + r^2p^2 = (pq + qr + rp)^2 - 2pqr(p + q + r) = s_2^2 - 2s_3s_1 = b^2 - 2ac$$

which wasn't so hard after all. Now you can solve the original question.

It turns out that *any* symmetric polynomial in p, q and r (a polynomial unchanged under all six permutations of p, q and r) can be expressed as a polynomial in s_1, s_2 and s_3. The generalization of this fact to n variables is known as the **fundamental theorem on symmetric polynomials**, which formed the starting point for Lagrange's *Réflexions sur la Résolution Algébrique des Equations*, which in turn paved the way for Evariste Galois' famous study of the same topic.

Arrangements for the Putnam

The exam will begin at **8 am** this Saturday. I have reserved room BH 217 for the exam and BH 201 for lunch. Please come to room BH 217 on Saturday no later than

`7:40 am`

Bring pencils/pens, an eraser and a watch. You are not allowed books, rulers or calculators. Cell phones must be turned off. I will provide scratch paper.