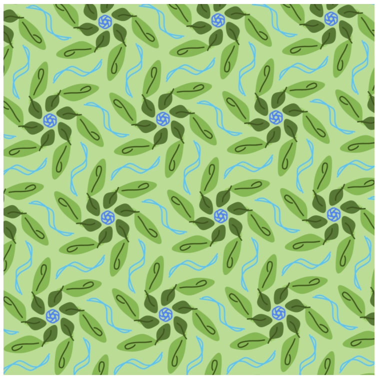
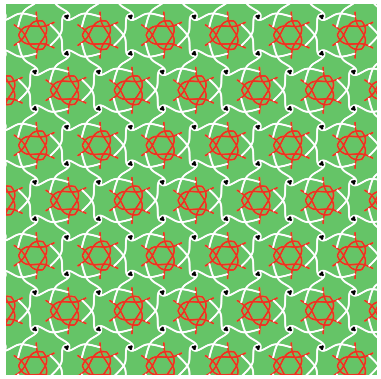
The Magic Theorem

by Chaim Goodman-Strauss, Heidi Burgiel, and John Horton Conway

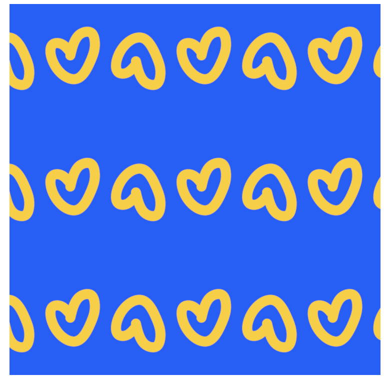
Reviewed by Dina Buric, Ian Caldwell, and Amites Sarkar



A 632 pattern by Dina Buric



A 3*3 pattern by Diana Sarkar



A xx pattern by Diana Sarkar

The central theme of this beautiful book is the classification of patterns: wallpaper patterns, spherical patterns, frieze patterns, and patterns in the hyperbolic plane. The classification is based on *signatures*; for instance, the signatures of the three patterns above are, from left to right, 632, 3*3, and xx. Given a pattern, one can find its signature by first identifying its lines of reflective symmetry (or *mirror lines*), then its centers of rotational symmetry (or *gyration points*), and, occasionally, a couple of other features (*miracles*, loosely associated with glide reflections, and *wonderings*, which appear in doubly periodic patterns). With a little practice, and a ruler, one can write down the signature of any pattern in a few minutes.

Each signature has a *cost*, which is the sum of the costs of its symbols. The symbols * and x both cost \$1, and a number n costs $\$(1/2)(1-1/n)$ if it appears after a * symbol, and $\$(1-1/n)$ otherwise. With these rules, the costs of the three signatures above are all \$2. The Magic Theorem of the title states that this is true for *all* wallpaper patterns: an easy consequence is that there are only 17 such patterns. (The theorem generalizes to other types of pattern.) The proof uses *orbifolds*, which are folded surfaces defined from patterns; the orbifolds of the patterns above are, from left to right, a triangular "pillowcase", a type of marked cone, and a Klein bottle.

The book is written in a very engaging way: drawing the reader in with a minimum of formalities, and inviting them to participate right from the start. It's purposefully structured so that useful results are presented long before their proofs. We recommend recording any unresolved points as you encounter them, and also skimming through any pages that feel confusing, and returning to them later. In particular, all three of us all found the initial descriptions of miracles and wonderings hard to follow.

The patterns above were generated using Jeff Weeks's KaleidoPaint software, which makes use of the signature notation from this book.

Dina: The Magic Theorem does an amazing job showcasing mathematical artists to bring many of its ideas to life. The planar patterns were created by software in collaboration with, among others, Troy Gilbert and Vladimir Bulatov. Bathsheba Grossman's metal sculptures are used to illustrate symmetries of spherical objects. The authors even present video games in hyperbolic space! Much of this art, such as Carolyn Yackel's tie dyed tessellations, could be used in the classroom. It was a pleasure to see and analyze so much mathematical art. The orbifold chapter was especially illustrative; seeing a tiling folded up into a torus or a Mobius strip really shows how mathematics can playfully build bridges, connecting seemingly distinct disciplines.

Ian: The aspects of The Magic Theorem that I found most valuable as a high school geometry teacher were the development of notation, the application to spheres, and the plentiful examples and guides for building classroom demos.

The notation is introduced in a way that would make sense to a math student familiar with symmetries and rigid transformations. It's a great example of how mathematicians interact with patterns: create a way to track, organize, and communicate about those patterns so they can further explore how patterns behave. Showing students how the ideas we explore in two dimensions can be extended to three dimensions motivates them to build stronger understandings in the simpler two dimensional context. The way The Magic Theorem works with symmetries on a sphere is not just accessible to high school students - they would find it fascinating.

Dina Buric is a software developer living in Bellingham, WA. She loves exploring the connections between math, art, and technology.

Ian Caldwell is a math teacher at Sehome High School in Bellingham, WA. Along with playing math with his Math Club friends, Ian runs a juggling club, knits, and explores how to make sounds on the mandolin.

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