Rainbow Turán problems for paths and forests of stars

Daniel Johnston
Department of Mathematical Sciences
University of Montana
Missoula, Montana 59812, U.S.A.
daniel1.johnston@umontana.edu

Cory Palmer*
Department of Mathematical Sciences
University of Montana
Missoula, Montana 59812, U.S.A.
cory.palmer@umontana.edu

Amites Sarkar
Department of Mathematics
Western Washington University
Bellingham, Washington 98225, U.S.A.
amites.sarkar@wwu.edu

Submitted: Sep 1, 2016; Accepted: Feb 3, 2017; Published: Feb 17, 2017
Mathematics Subject Classifications: 05C35, 05C15

Abstract

For a fixed graph F, we would like to determine the maximum number of edges in a properly edge-colored graph on n vertices which does not contain a *rainbow copy* of F, that is, a copy of F all of whose edges receive a different color. This maximum, denoted by $\mathrm{ex}^*(n, F)$, is the *rainbow Turán number* of F, and its systematic study was initiated by Keevash, Mubayi, Sudakov and Verstraëte [*Combinatorics, Probability and Computing* **16** (2007)]. We determine $\mathrm{ex}^*(n, F)$ exactly when F is a forest of stars, and give bounds on $\mathrm{ex}^*(n, F)$ when F is a path with l edges, disproving a conjecture in the aforementioned paper for $l = 4$.

1 Introduction

For a fixed graph F, we would like to determine the maximum number of edges in a properly edge-colored graph on n vertices which does not contain a *rainbow copy* of F, that is, a copy of F all of whose edges receive a different color. This maximum, denoted by $\mathrm{ex}^*(n, F)$, is the *rainbow Turán number* of F, and its systematic study was initiated by

*Research supported by University of Montana University Grant Program, grant no. M25364.

Keevash, Mubayi, Sudakov and Verstraëte in 2007 [12]. Among other things they proved that when F has chromatic number at least 3, then

$$\mathrm{ex}^*(n, F) = (1 + o(1))\mathrm{ex}(n, F)$$

where $\mathrm{ex}(n, F)$ is the (usual) Turán number of F. They also showed that

$$\mathrm{ex}^*(n, K_{s,t}) = O(n^{2-1/s})$$

where $K_{s,t}$ is the complete bipartite graph with classes of size s and t. This research was continued by Das, Lee and Sudakov [7], who partially answered a question from [12] on even cycles (this case has an interesting connection to additive number theory). In this paper, we determine $\mathrm{ex}^*(n, F)$ exactly when F is a forest of stars, and give bounds on $\mathrm{ex}^*(n, F)$ when F is a path with l edges, disproving a conjecture in [12] for $l = 4$.

Our methods also yield short proofs of the classic results on Erdős and Gallai on the (usual) Turán numbers of matchings [8], and of some recent results of Lidický, Liu and Palmer [13] on the Turán numbers of forests of stars. For all notation not defined see Bollobás [5].

2 Matchings

Write M_k for a matching with k edges. The usual Turán number for matchings was determined by Erdős and Gallai [8], who proved the following. Define $G_{n,k} = (V, E)$ to be the graph containing a clique G_k on vertex set $V_k \subset V$, where $|V| = n, |V_k| = k$, and in which each $v \in V_k$ is joined to every vertex of $W = V \setminus V_k$. Then

$$\begin{aligned} \mathrm{ex}(n, M_k) &= \max\{e(G_{n,k-1}), e(K_{2k-1})\} \\ &= \max\left\{\binom{k-1}{2} + (k-1)(n-k+1), \binom{2k-1}{2}\right\} \\ &= n(k-1) + O(k^2), \end{aligned}$$

and, for sufficiently large n, $G_{n,k-1}$ is the unique extremal graph. The second term of the maximum is necessary since a clique on $2k-1$ vertices also contains no M_k, and for small n it has more edges than $G_{n,k-1}$.

In other words, for sufficiently large n, $\mathrm{ex}(n, M_k) = \binom{k-1}{2} + (k-1)(n-k+1)$. Rather surprisingly, the same is true for $\mathrm{ex}^*(n, M_k)$. First we establish a weak version of this result. Although both the next two theorems are special cases of the results in the next section, their proofs will serve as templates for what follows.

Theorem 1.

$$\mathrm{ex}^*(n, M_k) = n(k-1) + O(k^2).$$

Proof. Suppose $G = (V, E)$ has the maximum number of edges such that there exists a proper edge-coloring χ of G with no rainbow M_k. Then G must contain a rainbow M_{k-1}, on vertex set A, say. Write $B = V \setminus A$, $C \subset A$ for those vertices of A which send at least $t = 2k$ edges to B, and set $c = |C|$.

We must have $c \leqslant k-1$, or else we could greedily build a rainbow matching from A to B of size k as follows. First choose an edge $c_1b_1 \in E$, where $c_1 \in C$ and $b_1 \in B$, where without loss of generality $\chi(c_1b_1) = 1$. Then choose an edge $c_2b_2 \in E$ of a different color, say $\chi(c_2b_2) = 2$, where $c_2 \in C$ and $b_2 \in B$ with $b_2 \neq b_1$. This is possible since $d(c_2) \geqslant 3$. Continuing, we finally choose $c_kb_k \in E$ with $\chi(c_kb_k) = k$, which is possible since $d(c_k) \geqslant 2k-1$ (we have $k-1$ vertices $b_1, \ldots, b_{k-1}$ and $k-1$ edge colors to avoid).

At least (and in fact, exactly) $k-1-c$ of the edges of our M_{k-1} contain no vertex of C; write M' for this set of edges. We claim that $G' = G[B]$ is $(k-1-c)$-colorable. Indeed, it is $(k-1-c)$-colored by χ. For if $e \in E(G')$ has a color not appearing among the colors of M', we can form a rainbow copy of M_k by starting with M' and e, and then greedily extending from the vertices of C as above (at the last stage we have $k-1$ colors and at most $(c-1)+2 \leqslant (k-2)+2 = k$ vertices to avoid). Consequently, the maximum degree in $G[B]$ is at most $k-1-c$, and so $e(G[B]) \leqslant \frac{k-1-c}{2}(n-(2k-2))$. Therefore,

$$\begin{aligned}
e(G) &= e(G[A]) + e(A,B) + e(G[B]) \\
&\leqslant \binom{2k-2}{2} + (2k-2-c)(2k-1) + c(n-(2k-2)) + \frac{k-1-c}{2}(n-(2k-2)) \\
&= (k-1)(6k-5) - c(2k-1) + \frac{k-1+c}{2}(n-(2k-2)) \\
&\leqslant (k-1)(6k-5) + (k-1)(n-(2k-2)) \\
&= n(k-1) + (k-1)(4k-3).
\end{aligned}$$

□

Next we refine this argument to get an exact result, at least for sufficiently large n.

Theorem 2. *For $n \geqslant 9k^2$,*

$$\mathrm{ex}^*(n, M_k) = \binom{k-1}{2} + (k-1)(n-k+1).$$

Proof. We already know that $\mathrm{ex}^*(n, M_k) \geqslant \mathrm{ex}(n, M_k) = \binom{k-1}{2} + (k-1)(n-k+1)$, so we only need to show that $\mathrm{ex}^*(n, M_k) \leqslant \binom{k-1}{2} + (k-1)(n-k+1)$. To this end, suppose again that $G = (V, E)$ has the maximum number of edges such that there exists a proper edge-coloring χ of G with no rainbow M_k. Following the proof of Theorem 1, we see that we must have $c = k-1$, since otherwise

$$e(G) \leqslant \frac{2k-3}{2}(n-2(k-1)) + (k-1)(6k-5) < \binom{k-1}{2} + (k-1)(n-k+1),$$

as long as $n \geqslant 9k^2$. Armed with this information, we deduce that $G[(A \cup B) \setminus C]$ contains no edges. Otherwise, if $e \in E(G[(A \cup B) \setminus C])$, we could greedily extend e to a rainbow matching M_k using the vertices of C. Consequently,

$$e(G) \leqslant \binom{|C|}{2} + |C|(|A| - |C| + |B|) = \binom{k-1}{2} + (k-1)(n-k+1).$$

□

The theorem of Erdős and Gallai that $\mathrm{ex}(n, M_k) = \binom{k-1}{2} + (k-1)(n-k+1)$ follows immediately from Theorem 2 (at least for sufficiently large n)[1].

3 Forests of stars

In this section we address the rainbow Turán number of a forest F where each component is a star. In this case, the Turán number was determined by Lidický, Liu and Palmer [13]. We give a new proof of this result at the end of this section.

Let F be a forest of k stars $S_1, S_2, \ldots, S_k$ such that $e(S_j) \leqslant e(S_{j+1})$ for each j. We will construct a family of n-vertex graphs that each have a proper edge-coloring with no rainbow copy of F. For $0 \leqslant c \leqslant k-1$, define $f(c)$ to be

$$f(c) = \left(\sum_{i=1}^{k-c} e(S_i)\right) - 1.$$

The graph $H_F(n, c)$ is defined as follows. For $c = k-1$, we connect a set C of $c = k-1$ universal vertices to an edge-maximal graph H of maximum degree $f(c) = f(k-1) = e(S_1) - 1$ on the remaining $n-k+1$ vertices. (A universal vertex is one that is joined to every other vertex, so that in particular $G[C]$ is a clique.) When $c \leqslant k-2$, we connect a set C of c universal vertices to an edge-maximal $f(c)$-edge-colorable graph H on $n-c$ vertices.

Note the slight distinction in the definition of the subgraph H in the two cases $c = k-1$ and $c \leqslant k-2$. In both cases, it is easy to see that H can only contain $k-c-1$ of the stars in F. The remaining $c+1$ stars must each use at least one vertex from C, which is impossible. Therefore, in both cases, $H_F(n, c)$ does not contain a rainbow copy of F.

When $c = k-1$, the subgraph H is $(e(S_1)-1)$-regular when either $n-c$ or $e(S_1)-1$ is even. Otherwise, H has one vertex of degree $e(S_1)-2$ and $n-k$ vertices of degree $e(S_1)-1$. Therefore, the total number of edges in $H_F(n, k-1)$ is

$$e(H_F(n, k-1)) = \binom{k-1}{2} + (k-1)(n-k+1) + \left\lfloor \frac{(e(S_1)-1)(n-k+1)}{2} \right\rfloor.$$

When $c \leqslant k-2$, there are exactly $\lfloor \frac{n-c}{2} \rfloor$ edges of each color in H, so that H has $f(c)\lfloor \frac{n-c}{2} \rfloor$ edges. Therefore, the total number of edges in $H_F(n, c)$ is

$$\begin{aligned} e(H_F(n, c)) &= \binom{c}{2} + c(n-c) + f(c)\left\lfloor \frac{n-c}{2} \right\rfloor \\ &= \binom{c}{2} + c(n-c) + \left(\left(\sum_{i=1}^{k-c} e(S_i)\right) - 1\right)\left\lfloor \frac{n-c}{2} \right\rfloor. \end{aligned}$$

[1]In fact, to get a short direct proof of the theorem of Erdős and Gallai simply remove all reference to edge-colorings in the argument above. Note that this proof avoids Hall's theorem.

Consequently, for all $c \leqslant k-1$, the number of edges in the graph $H_F(n,c)$ is

$$e(H_F(n,c)) = cn + \frac{1}{2}\left(\left(\sum_{i=1}^{k-c} e(S_i)\right) - 1\right)n + O(1). \tag{1}$$

Furthermore, the subgraph H of $H_F(n,c)$ has average degree $f(c) - \epsilon$, where $\epsilon < 1$.

Of particular interest is the construction $H_F(n,0)$, which is simply an edge-maximal $(e(F)-1)$-edge-colored graph, since $f(0) = e(F) - 1$.

The key to our analysis is the following technical lemma, which allows us to restrict our attention to the family $H_F(n,c)$.

Lemma 3. *Let F be a forest of k stars. Suppose that G is an edge-maximal properly edge-colored graph on n vertices containing no rainbow copy of F. Then, for sufficiently large n, G is isomorphic to one of the graphs $H_F(n,c)$.*

Before turning to the proof of this lemma, we explain its use in the proof of our main result, Theorem 4. Specifically, suppose we have proved Lemma 3, and consider a fixed forest of stars F. In order to find the extremal graphs for a rainbow copy of F, we just need to determine the value of $c = c(F)$ that maximizes the number of edges $e(H_F(n,c))$ of $H_F(n,c)$.

For example, when F is a forest of stars each of size 1 (i.e., a matching), then, for large n, the sum in (1) is maximized when $c = k-1$. Therefore, for large n, an edge-maximal properly edge-colored graph G containing no rainbow copy of F must be isomorphic to $H_F(n,k-1)$. In this case, $f(k-1) = e(S_1) - 1 = 0$ (this holds whenever F contains a star of size 1), so that G consists of a universal set of size $k-1$ joined to an independent set of size $n-k+1$. This reproves Theorem 2.

It turns out that, for every F, the maximum of $e(H_F(n,c))$ is attained at either $c = 0$ or $c = k-1$.

Theorem 4. *Let F be a forest of k stars. Suppose that G is an edge-maximal properly edge-colored graph on n vertices containing no rainbow copy of F. Then, for sufficiently large n, 1) if F contains no star of size 1, then G is isomorphic to $H_F(n,0)$;*
2) otherwise, G is isomorphic to the larger of $H_F(n,0)$ and $H_F(n,k-1)$.

Proof. First consider the case when F contains no star of size 1. In this case, if F contains at least one star of size at least 3, then, for sufficiently large n, the right hand side of (1) is maximized when $c = 0$. Therefore, by Lemma 3, G must be isomorphic to $H_F(n,0)$ (for large n).

If every star in F has size 2, then the sum of the two main terms in (1) is constant over all $c \leqslant k-1$, so we need to examine the error term. In both the cases $c = k-1$ and $c \leqslant k-2$, we have

$$e(H_F(n,c)) = \binom{c}{2} + c(n-c) + (2(k-c)-1)\left\lfloor \frac{n-c}{2} \right\rfloor.$$

Simple computations show that this is maximized at $c = 0$. Therefore, G must be isomorphic to $H_F(n, 0)$.

To summarize, if F contains no star of size 1, G must be isomorphic to $H_F(n, 0)$, if n is sufficiently large. As already mentioned, this extremal graph is just an edge-maximal graph that is properly edge-colored with $f(0) = e(F) - 1$ colors.

Now suppose that F contains a star of size 1. Write $s \geqslant 1$ for the number of stars of size 1, t for the number of stars of size 2, and $p = k - s - t$ for the number of stars of size at least 3 in F. If $p = 0$, then we should clearly take $c = k - 1$ to maximize the sum of the two main terms in (1). Consequently, we may assume $p > 0$. We now have three estimates for the number of edges in $H_F(n, c)$, depending on the value of c. If $c < p$ (and $p > 0$), then

$$e(H_F(n, c)) = cn + \frac{1}{2}\left(s + 2t + \left(\sum_{i=s+t+1}^{k-c} e(S_i)\right) - 1\right) n + O(1),$$

which is maximized (for large n) when $c = 0$ (as each $e(S_i)$ in the above sum is at least 3). Thus, when $c < p$ (and $p > 0$), we should take $c = 0$, and then

$$e(H_F(n, c)) = \frac{1}{2}\left(s + 2t + \left(\sum_{i=s+t+1}^{k} e(S_i)\right) - 1\right) n + O(1). \tag{2}$$

If next $p \leqslant c < p + t$, then

$$e(H_F(n, c)) = cn + \frac{1}{2}(s + 2(t - (c - p)) - 1)n + O(1) = \frac{1}{2}(s + 2t + 2p - 1)n + O(1), \tag{3}$$

which (for large n) is clearly smaller than (2) if $p > 0$. If lastly $p + t \leqslant c \leqslant p + t + s - 1 = k - 1$, then

$$e(H_F(n, c)) = cn + \frac{1}{2}(s - (c - (p + t)) - 1)n + O(1) = \frac{1}{2}(s + t + p + c - 1)n + O(1),$$

which is maximized (for large n) when $c = k - 1$. (We remind the reader that in the case we are considering, $f(k - 1) = e(S_1) - 1 = 0$, so that both constructions of $H_F(n, c)$ coincide when $c = k - 1$.) Thus, when $p + t \leqslant c \leqslant p + t + s - 1 = k - 1$, we should take $c = k - 1 = s + t + p - 1$, and then

$$e(H_F(n, c)) = (s + t + p - 1)n + O(1) = (k - 1)n + O(1),$$

which is larger than (3) when n is large. Therefore, for sufficiently large n, the number of edges in $H_F(n, c)$ is maximized when c is either 0 or $k - 1$. □

The choice of c to maximize the sum of the two main terms in (1) can be illustrated as follows (see Table 1). Write down a row of k 2s, and underneath this row, write down the star sizes $e(S_k), e(S_{k-1}), \ldots, e(S_1)$ in decreasing order. Next, take the sum of the first c entries in the top row and the last $k - c$ entries in the bottom row, where $c \leqslant k - 1$. This sum represents twice the coefficient of n in (1).

We now turn our attention to the proof of Lemma 3. We begin with a simple lemma.

p					t					s				
2	**2**	**2**	**2**	2	2	2	2	2	2	2	2	2	2	2
5	4	4	3	**3**	**2**	**2**	**2**	**2**	**2**	**1**	**1**	**1**	**1**	**1**

Table 1: Illustration of the proof of Theorem 4

Lemma 5. *Fix positive integers d and Δ and a constant $0 \leqslant \epsilon < 1$. If G is a graph with average degree at least $d - \epsilon$ and maximum degree at most Δ, then the number of vertices in G of degree less than d is at most*

$$\frac{\Delta - d + \epsilon}{\Delta - d + 1} n.$$

In particular, the number of vertices in G of degree at least d is $\Omega(n)$ (i.e. at least Cn where $C = C(d, \Delta, \epsilon) > 0$).

Proof. The sum of the degrees in G is at least $(d - \epsilon)n$. On the other hand, if x is the number of vertices of degree less than d in G, then the sum of the degrees in G is at most

$$(d - 1)x + \Delta(n - x).$$

Combining these two estimates and solving for x gives the result. □

We are now ready to prove Lemma 3.

Proof of Lemma 3. Let G be as in the statement of the theorem, and let C be the set of vertices in G of degree at least $3e(F)$. Write $c = |C|$. Observe that $c \leqslant k - 1$, since otherwise we could greedily embed the components of F into G, using the vertices of C as their centers.

The subgraph $G' = G[V \setminus C]$ has maximum degree at most $3e(F)$. Since G has at least as many edges as the graph $H_F(n, c)$, it follows that G' must have average degree at least $f(c) - \epsilon$, for some $\epsilon < 1$. Therefore, by Lemma 5, the subgraph G' has at least $\Omega(n)$ vertices of degree

$$f(c) = \left(\sum_{i=1}^{k-c} e(S_i) \right) - 1.$$

Now suppose (for a contradiction) that G' has a vertex v of degree greater than $f(c)$. Then we can form a rainbow copy of F in G as follows. Choose $k - c - 1$ vertices of G' of degree $f(c)$ that are at distance at least 3 from each other and from v (this is possible since the maximum degree is constant). We can build a rainbow forest of the stars $S_1, S_2, \ldots, S_{k-c-1}$ on these vertices, since these stars use $f(c) + 1 - e(S_{k-c})$ edge colors. The vertex v has degree at least $f(c) + 1$, so it is incident to at least $f(c) + 1 - (f(c) + 1 - e(S_{k-c})) = e(S_{k-c})$ unused colors. Therefore, we can extend the rainbow forest to include S_{k-c}. Finally, the remaining c stars of F can be greedily embedded using the vertices in C as their centers, so that G contains a rainbow copy of F.

This is a contradiction. Therefore, G' has maximum degree at most $f(c)$. When $c = k-1$ we are done, since we have shown that G has at most as many edges as $H_F(n, k-1)$.

Let us now consider the case $c \leqslant k-2$. Recall that by its construction if we remove the set of c universal vertices (i.e. vertices of degree $n-1$) from $H_F(n,c)$, then we are left with an edge-maximal $f(c)$-edge-colorable graph H on $n-c$ vertices (see the construction at the beginning of this section). On the other hand, we remove c vertices of degree at most $n-1$ from G to get G'. Therefore, as $e(G) \geqslant e(H_F(n,c))$ we have that the number of edges in G' is at least the number of edges in H. Thus,

$$e(G') \geqslant e(H) = f(c)\left\lfloor \frac{n-c}{2} \right\rfloor \geqslant f(c)\left(\frac{n-c}{2}\right) - \left\lfloor \frac{f(c)}{2} \right\rfloor. \tag{4}$$

In particular, G' has $n - O(1)$ vertices of degree $f(c)$, since G' has maximum degree $f(c)$. We claim that G' must be colored with $f(c)$ edge colors. Suppose, for a contradiction, that G' is colored with at least $f(c)+1$ colors. Then there is a color class, say *red*, with at most

$$\frac{1}{f(c)+1}\left\lfloor \frac{n-c}{2} \right\rfloor$$

edges. Therefore, there are $\Omega(n)$ vertices in G' of degree $f(c)$ that are not incident to a red edge.

Since $c \leqslant k-2$, the sum in $f(c)$ has at least two terms, so that

$$2e(S_1) \leqslant e(S_1) + e(S_2) \leqslant \sum_{i=1}^{k-c} e(S_i) = f(c) + 1.$$

As $e(S_1)$ is an integer, this implies that $e(S_1) \leqslant \lceil f(c)/2 \rceil$.

We now embed S_1 in G' using a red edge. If $n-c$ is even, then by (4) and the fact that G' has maximum degree at most $f(c)$, we have that every vertex in G' has degree $f(c)$. As $f(c) \geqslant \lceil f(c)/2 \rceil \geqslant e(S_1)$, we can choose a vertex v incident to a red edge and embed S_1 using that red edge.

When $n-c$ is odd, G' may contain vertices of degree less than $f(c)$. Consider a red edge uv and observe that at least one of the vertices u and v (say v) has degree at least $\lceil f(c)/2 \rceil$; otherwise the number of edges in G' is less than $f(c)\left\lfloor \frac{n-c}{2} \right\rfloor$. Therefore, we can embed S_1 using the red edge uv with v as the center.

Now, among the vertices not incident to red edges, pick $k-c-1$ vertices of degree $f(c)$ that are at distance at least 3 from each other and from the center v of S_1. Using these vertices as centers, we can greedily build a rainbow forest of stars $S_2, S_3, \dots, S_{k-c}$, since we have only used at most $e(S_1)-1$ of the $f(c)$ colors incident to these vertices. Finally, the remaining c stars of F can be greedily embedded using the vertices in C as their centers, so that G contains a rainbow copy of F. This is a contradiction. Therefore, G' is properly $f(c)$-edge-colored. □

We now give a new proof of the result of Lidický, Liu and Palmer on the Turán number of forests of stars.

We begin by describing the extremal graph for the forest of stars $S_1, S_2, \ldots, S_k$, where $e(S_j) \leqslant e(S_{j+1})$ for each j. Let $H'_F(n,i)$ be the graph obtained by connecting a set of i universal vertices to an edge-maximal graph of maximal degree $e(S_{k-i})-1$ on $n-i$ vertices. Observe that if one of $e(S_{k-i})-1$ or $n-i$ is even, and n is large enough, then H is $(e(S_{k-i})-1)$-regular. If both are odd, then H has exactly one vertex of degree $e(S_{k-i})-2$, and $n-i-1$ vertices of degree $e(S_{k-i})-1$. Each of the graphs $H'_F(n,i)$ is F-free, since otherwise each of the $i+1$ stars $S_k, S_{k-1}, \ldots, S_{k-i}$ must use at least one vertex from the universal set of size i, which is impossible.

Theorem 6 (Lidický, Liu, Palmer [13])**.** *Let F be a forest of k stars $S_1, S_2, \ldots, S_k$, such that $e(S_j) \leqslant e(S_{j+1})$ for each j. Then*

$$\mathrm{ex}(n,F) = \max_{0 \leqslant i \leqslant k-1} \left\{ i(n-i) + \binom{i}{2} + \left\lfloor \frac{(e(S_{k-i})-1)(n-i)}{2} \right\rfloor \right\}.$$

Proof. Note that G has at least as many edges as $H'_F(n,i)$ for all $i \leqslant k-1$. Suppose that G has a set C of c vertices of degree at least $e(F)$. We must have $c \leqslant k-1$, since otherwise we could greedily embed F from the vertices of C. Let $G' = G[V \setminus C]$ be the graph on the remaining $n-c$ vertices. The maximum degree of G' is less than $e(F)$. First let us suppose that $c = k-1$. In this case, we claim that the maximum degree of G' is at most $e(S_1)-1$. Indeed, if there is a vertex v of higher degree, then we can embed S_1 into G' using v, and complete the forest F by greedily embedding the stars $S_2, S_3, \ldots S_k$ using the vertices of C as their centers.

Next suppose that $c < k-1$. Suppose (for a contradiction) that $e(S_{k-c-1}) = e(S_{k-c})$. Comparing G to $H'_F(n, c+1)$, we see that G' must have average degree at least $e(S_{k-c-1}) - \epsilon = e(S_{k-c}) - \epsilon$. Therefore, by Lemma 5, the graph G' contains $\Omega(n)$ vertices of degree at least $e(S_{k-c})$. Now we can embed F as follows. Choose $k-c$ vertices of G' of degree $e(S_{k-c})$ that are at distance at least 3 from each other. We can embed the stars $S_1, S_2, \ldots, S_{k-c}$ on these vertices. Next we can greedily embed the remaining stars $S_{k-c+1}, \ldots, S_k$ into G using the vertices of C as their centers; a contradiction.

Therefore, we may assume that $e(S_{k-c-1}) < e(S_{k-c})$. By comparing G to $H'_F(n,c)$, we see that G' must have average degree at least $e(S_{k-c})-1$. Therefore, by Lemma 5, the graph G' contains $\Omega(n)$ vertices of degree at least $e(S_{k-c})-1$. Now suppose that G' has a vertex v of degree greater than $e(S_{k-c})-1$. Then we can embed F as follows. Choose $k-c-1$ vertices of G' of degree $e(S_{k-c})-1$ that are at distance at least 3 from each other and from v. We can embed the stars $S_1, S_2, \ldots, S_{k-c-1}$ on these vertices, since $e(S_{k-c})-1 \geqslant e(S_{k-c-1})$. Next we embed the star S_{k-c} at v, and then greedily embed the remaining stars $S_{k-c+1}, \ldots, S_k$ into G using the vertices of C as their centers; a contradiction. Therefore, the maximum degree of G' is $e(S_{k-c})-1$. □

4 Paths

In this paper, P_l will denote a path with l *edges*, which we will call a path of length l. The usual Turán number for paths was determined asymptotically by Erdős and Gallai [8], and exactly by Faudree and Schelp [9]. Erdős and Gallai proved that, given a path length l, if l divides n then

$$\mathrm{ex}(n, P_l) = \frac{n}{l}\binom{l}{2} = \frac{l-1}{2}n,$$

and the unique extremal graph is the disjoint union of $\frac{n}{l}$ copies of K_l. We briefly recall the proof. First we show that any graph G with minimum degree at least δ contains a path of length 2δ (provided of course that $2\delta < n$). Next, consider a graph G of order n with more than $\frac{l-1}{2}n$ edges (i.e., of average degree greater than $l-1$). By repeatedly removing a vertex of minimum degree, we can show that G must contain a subgraph H whose minimum degree is at least $\frac{l}{2}$, and so H contains a path of length l.

Following this approach for the rainbow Turán problem therefore requires us to find a *rainbow* path of length $c\delta$ in a graph of minimum degree δ. To this end, we have the following theorem, which generalizes a result of Gyárfás and Mhalla [11], and is itself a special case of a theorem of Babu, Chandran and Rajendraprasad [3]. For completeness, we provide a short proof of the result we need, which is less technical than the proof in [3].

Theorem 7. *Let G be a graph with minimum degree $\delta = \delta(G)$. Then any proper edge-coloring of G contains a rainbow path of length at least $\frac{2}{3}\delta$.*

Proof. Suppose that c is a proper edge-coloring of G. Take a longest rainbow path $P = v_0v_1\cdots v_l$ in G, of length l. Without loss of generality, $c(v_{i-1}v_i) = i$ for each i (i.e., the i^{th} edge of P receives color i). Write s_o for the number of edges colored with colors $1, \dots, l$ that v_0 sends to vertices outside P, and note that v_0 can send no other edges outside P, or else P could be extended. Also write s_i for the number of edges of colors $1, \dots, l$ that v_0 sends to other vertices of P (including v_1), and write $s^\times$ for the number of edges of other colors that v_0 sends to vertices of P. Finally, define t_o, t_i and $t^\times$ to be the analogous quantities for v_l.

Observe now that

$$s_o + s_i \leqslant l, \tag{1}$$

since c is a proper coloring, that

$$s_i + s^\times \leqslant l, \tag{2}$$

since there are exactly l vertices on P other than v_0, and that

$$s_o + t^\times \leqslant l, \tag{3}$$

since if $v_iv_l \in E(G)$ with $c(v_iv_l) > l$ then there is no $w \notin V(P)$ with $c(wv_0) = c(v_iv_{i+1}) = i+1$, or else $wv_0v_1\cdots v_iv_lv_{l-1}\cdots v_{i+1}$ would be a rainbow path in G of length $l+1$. Analogous inequalities hold for t_o, t_i and $t^\times$.

Consequently, combining (1), (2) and (3) with the minimum degree condition, we have

$$2\delta \leqslant (s_o + s_i + s^\times) + (t_o + t_i + t^\times) = (s_i + s^\times) + (s_o + t^\times) + (t_o + t_i) \leqslant l + l + l = 3l,$$

so that $l \geqslant \frac{2}{3}\delta$, as desired. □

We remark that the constant $\frac{2}{3}$ cannot be improved in general. To see this, let G be the disjoint union of r copies of K_4, and properly 3-color the edges of each K_4 (there is a unique way to do this, up to isomorphism). Then $\delta(G) = 3$, and the longest rainbow path in G has length 2. However, when considering complete graphs, Alon, Pokrovskiy and Sudakov [1] proved that a proper edge-coloring of K_n contains a rainbow cycle of length $n - o(n)$ (improving the bound $\frac{3}{4}n - o(n)$ by Chen and Li [6], and independently Gebauer and Mousset [10]). On the other hand, Maamoun and Meyniel [14] showed that we are not always guaranteed a rainbow path of length $n - 1$. In their construction, $n = 2^k$, and we identify the vertices of K_{2^k} with the points of the Boolean cube $\{0, 1\}^k$. If we now color each edge $\mathbf{uv}$ with color $\mathbf{u} - \mathbf{v} \neq \mathbf{0}$, a monochromatic path $\mathbf{v_0 v_1} \cdots \mathbf{v_{n-1}}$ of length $n - 1$ in K_n would involve all possible colors (except for $\mathbf{0}$), so that

$$\mathbf{v_0} - \mathbf{v_{n-1}} = \sum_{i=0}^{n-2} (\mathbf{v_i} - \mathbf{v_{i+1}}) = \sum_{\mathbf{0} \neq \mathbf{x} \in \{0,1\}^k} \mathbf{x} = \sum_{\mathbf{x} \in \{0,1\}^k} \mathbf{x} = \mathbf{0},$$

which implies that $v_0 = v_{n-1}$, a contradiction.

A slight modification of the proof of Theorem 7 yields a short proof of the full result of Babu, Chandran and Rajendraprasad [3] mentioned above. Their result deals with general (not necessarily proper) edge-colorings, in which, given an edge-colored graph G, $\theta(G)$ is the minimum number of distinct colors seen at each vertex. Clearly $\theta(G) = \delta(G)$ if the coloring is proper.

Theorem 8. *Let G be an edge-colored graph in which every vertex is incident to at least $\theta = \theta(G)$ edge-colors. Then G contains a rainbow path of length at least $\frac{2}{3}\theta$.*

Proof. We follow the proof of Theorem 7, with a slight change in the definitions of s_o, s_i and $s^\times$. This time, s_o is the number of *colors* of edges that v_0 sends to vertices outside P (as before, each of these colors already occurs on P), and $s^\times$ is the number of colors not seen on P which occur as the colors of edges v_0 sends to P. Now s_i is the number of colors from 1 to l that occur as colors of edges v_0 sends to P and *which are not counted in s_o*. The rest of the proof goes through as before, with δ replaced by θ. □

Returning to the problem at hand, we can use Theorem 7 to obtain a bound on the rainbow Turán number of paths.

Theorem 9. *For each fixed $l \geqslant 1$, we have*

$$\frac{l-1}{2} n \sim \mathrm{ex}(n, P_l) \leqslant \mathrm{ex}^*(n, P_l) \leqslant \left\lceil \frac{3l-2}{2} \right\rceil n.$$

Proof. We will make use of the standard fact that a graph G of average degree more than $2d$ contains a subgraph H of minimum degree at least $d+1$. This is proved by repeatedly removing a vertex of minimum degree from G.

First, suppose that l is even, and write $l = 2k$. Let G be a graph of order n with more than $\frac{3l-2}{2} n = (3k-1)n$ edges (and so of average degree more than $2(3k-1)$). Then G

contains a subgraph H of minimum degree at least $3k$, which by Theorem 7 contains a rainbow path of length $2k = l$.

Second, suppose that l is odd, and write $l = 2k + 1$. Let G be a graph of order n with more than $\frac{3l-1}{2} = (3k + 1)n$ edges (and so of average degree more than $2(3k + 1)$). Then G contains a subgraph H of minimum degree at least $3k + 2$, which by Theorem 7 contains a rainbow path of length $2k + 1 = l$. □

For small values of l, one can do considerably better. It is trivial that $\mathrm{ex}^*(n, P_1) = \mathrm{ex}(n, P_1) = 0$ and that $\mathrm{ex}^*(n, P_2) = \mathrm{ex}(n, P_2) = \lfloor \frac{n}{2} \rfloor$. When $l = 3$, we have the following simple result.

Theorem 10. *Suppose that* n *is divisible by* 4*. Then* $\mathrm{ex}^*(n, P_3) = \frac{3n}{2} = \frac{3}{2}\mathrm{ex}(n, P_3) + O(1)$.

Proof. The example already shown, namely $\frac{n}{4}$ disjoint copies of properly 3-colored K_4s, shows that $\mathrm{ex}^*(n, P_3) \geqslant \frac{3n}{2}$. For the other direction, suppose that $G = (V, E)$ is a graph with more than $\frac{3n}{2}$ edges and no rainbow P_3, and select $v \in V$ with $d(v) \geqslant 3$ (there must be at least one such v). Then the neighbors $v_1, \ldots, v_r$ of v can only be adjacent to each other, since if $v_i w \in E$ with $vw \notin E$ then wv_ivv_j is a rainbow P_3 for some j (chosen so that the colors of v_iw and vv_j are different). Moreover, if $d(v) \geqslant 4$, then $G[v \cup \Gamma(v)]$ is a star, since if $v_iv_j \in E$ then $v_jv_ivv_k$ is a rainbow P_3, where this time k has been chosen so that v_iv_j and vv_k receive different colors. Consequently, if $d(v) \geqslant 3$, then $G_v = G[v \cup \Gamma(v)]$ is a component of G whose average degree is at most 3, so we may remove it and apply induction. □

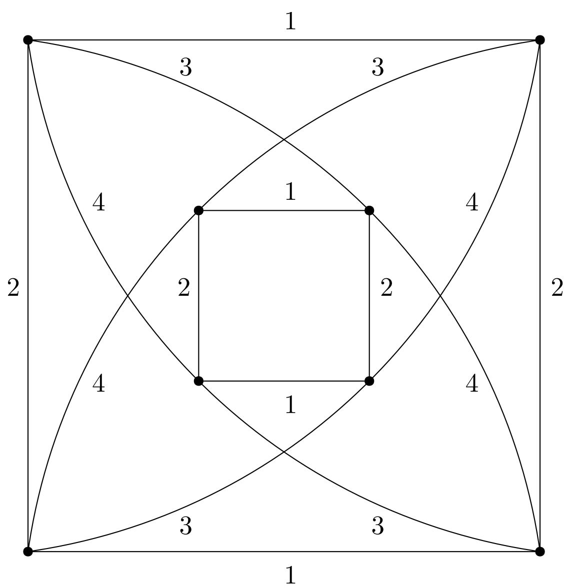

Figure 1: A proper edge-coloring of $K_{4,4}$ with no rainbow P_4

For P_4, we have the following theorem.

Theorem 11. *If n is divisible by 8, then* $\mathrm{ex}^*(n, P_4) = 2n$*. In general,* $\mathrm{ex}^*(n, P_4) = 2n + O(1)$*.*

Proof. The lower bound comes from the proper edge-coloring of $K_{4,4}$ illustrated in Figure 1, which contains no rainbow P_4. (To see this, note that in the given coloring, any 4-cycle containing two identically-colored edges must in fact be 2-colored, so that every 4-cycle contains either 2 or 4 colors. Now suppose (to the contrary) that $xyzst$ is a rainbow P_4. Then the cycle $xyzsx$ must contain all 4 colors, so that edges st and sx must receive the same color, which is impossible since they are adjacent.) Next, if $n = 8k$, then the disjoint union of k such edge-colored $K_{4,4}$s has $2n$ edges and no rainbow P_4. Consequently, $\mathrm{ex}^*(n, P_4) \geqslant 2n$ if $8|n$, and $\mathrm{ex}^*(n, P_4) \geqslant 2n + O(1)$ in general.

For the upper bound, we show that every proper edge-coloring of an n-vertex graph G with $m > 2n$ edges contains a rainbow P_4.

As noted before, G contains a subgraph G' of minimum degree at least 3, since otherwise we can repeatedly remove vertices of degrees 1 and 2 so that the average degree increases. Furthermore, G' has average degree greater than 4. Therefore, G' has a vertex v of degree at least 5. We will show that G' contains a rainbow P_4. The proof now splits into two cases.

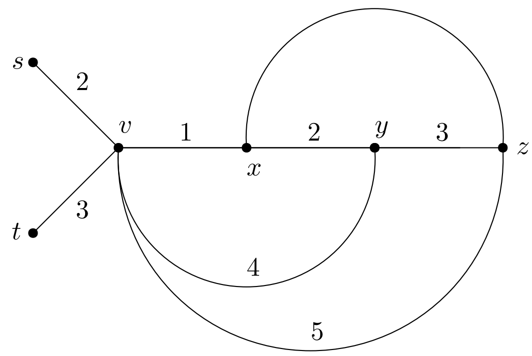

Figure 2: A rainbow P_3 ending at a vertex v of degree at least 5

Case 1: G' contains a rainbow P_3 ending at v. This case is illustrated in Figure 2; let the rainbow P_3 be $P = vxyz$, where edges vx, xy and yz are colored 1, 2 and 3 respectively. Since v has degree at least 5, it must be adjacent to at least 2 vertices not on P; suppose these vertices are s and t. If either of the edges vs and vt receives a color other than 2 or 3, then we have a rainbow P_4. Now suppose that $c(vs) = 2$ and $c(vt) = 3$, where c denotes the color of the edge. If v is adjacent to any other vertex u not on P, then since $c(uv)$ would have to be different from 1, 2 and 3, the edge uv with P forms a rainbow P_4. Otherwise, the vertex v has degree 5 and is adjacent to both y and z. Without loss of generality, suppose $c(vy) = 4$ and $c(vz) = 5$.

Suppose that the vertex z is adjacent to x. Note that $c(xz)$ cannot be 1, 2 or 3, and so $svxzy$ is a rainbow P_4. If z is not adjacent to x, then z is adjacent to a vertex w not on P (possibly $w = s$ or $w = t$) as the minimum degree of G' is at least 3. We know that $c(wz)$ cannot be 3 or 5; if $c(wz) = 1$ then $wzvyx$ is a rainbow P_4, while if $c(wz) = 2$ then

$wzyvx$ is a rainbow P_4. However, if $c(wz)$ is not 1, 2 or 3, then $vxyzw$ is a rainbow P_4. Accordingly, this completes the proof in Case 1.

Case 2: G' contains no rainbow P_3 ending at v. Since $\delta(G') \geqslant 3$, G' contains a rainbow P_2 ending at v; let this path be vxy, where $c(vx) = 1$ and $c(xy) = 2$. The vertex y has degree at least 3; if y were adjacent to two vertices s and t other than v and x, then one of edges ys and yt would receive color 3, creating a rainbow P_3 ending at v. Consequently, the degree of y is 3 and y is adjacent to v and a new vertex z. Furthermore, $c(yz) = 1$, and, without loss of generality, $c(yv) = 3$. Let P be the path $vxyz$.

The vertex z is adjacent to at most one vertex w not on P and the edge zw must receive color 3 to avoid the rainbow P_3 $vyzw$ ending at v. Consequently, z is adjacent to at least one of v or x. The proof now splits into three sub-cases.

Case 2A: z is adjacent to x and a new vertex w. This case is illustrated on the left of Figure 3. Edge xz cannot receive any of colors 1, 2 or 3, and so $vxzw$ is a rainbow P_3 ending at v.

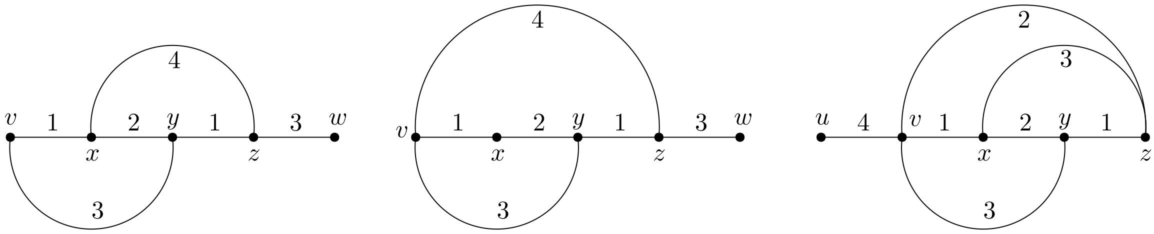

Figure 3: No rainbow P_3 ends at a vertex v of degree at least 5

Case 2B: z is adjacent to v and a new vertex w. This case is illustrated in the center of Figure 3. Edge vz must receive color 2 to avoid the rainbow P_3 $vzyx$ ending at v. Now, if w were adjacent to two vertices s and t other than v, x, y and z, then one of edges ws and wt would receive color other than 2 and 3, creating a rainbow P_3 ending at v. Therefore, there is at least one edge from w to v, x, or y. Such an edge cannot receive colors 1, 2, or 3. If wv is an edge, then $vwzy$ is a rainbow P_3; if wx is an edge, then $vxwz$ is a rainbow P_3; if wy is an edge, then $vxyw$ is a rainbow P_3. In all cases we have found a rainbow P_3 ending at v.

Case 2C: z is adjacent to both v and x. This case is illustrated on the right of Figure 3. In this case, the vertices v, x, y, z induce a properly 3-edge-colored K_4 as otherwise we can easily find a rainbow P_3 ending at v. We will exploit the resulting symmetry in the three colors 1, 2 and 3. The vertex v must be adjacent to a new vertex u, and, without loss of generality, $c(uv) = 4$. If the vertex u is adjacent to a new vertex w, then we may assume that $c(uw) = 1$, and then $wuvzx$ would be a rainbow P_4. Otherwise, u is adjacent to at least two of x, y and z; suppose it is adjacent to x. Then $c(ux)$ cannot be 1, 2, 3 or 4, and then $xuvzy$ is a rainbow P_4.

Thus, in all three sub-cases we obtain either a rainbow P_3 ending at v (leading us to Case 1), or a rainbow P_4 in G'. □

Keevash, Mubayi, Sudakov and Verstraëte conjectured that the extremal example for rainbow P_ls is a disjoint union of cliques of size $c(l)$, where $c(l)$ is chosen as large as possible so that $K_{c(l)}$ can be properly edge-colored with no rainbow P_l. It is not hard to show that a properly edge-colored K_5 must contain a rainbow P_4, so that $c(4) = 4$. Consequently, the conjecture implies that $\mathrm{ex}^*(n, P_4) = \frac{3n}{2} + O(1)$, which is false, as our theorem shows. However, we note that the conjecture may still hold for longer paths.

References

[1] N. Alon, A. Pokrovskiy, B. Sudakov, Random subgraphs of properly edge-coloured complete graphs and long rainbow cycles, arXiv:1608.07028 (2016).

[2] L. Andersen, Hamilton circuits with many colours in properly edge-coloured complete graphs, *Mathematica Scandinavica* **64** (1989), 5–14.

[3] J. Babu, L. Sunil Chandran and D. Rajendraprasad, Heterochromatic paths in edge colored graphs without small cycles and heterochromatic-triangle-free graphs, *European Journal of Combinatorics* **48** (2015), 110–126.

[4] N. Bushaw and N. Kettle, Turán numbers of multiple paths and equibipartite trees, *Combinatorics, Probability and Computing* **20** (2011), 837–853.

[5] B. Bollobás. *Modern Graph Theory* (3rd ed.), Graduate Texts in Mathematics **184**, Springer, USA, 1998.

[6] H. Chen and X. Li, Long rainbow path in properly edge-colored complete graphs, arXiv:1503.04516 (2015).

[7] S. Das, C. Lee and B. Sudakov, Rainbow Turán problem for even cycles, *European Journal of Combinatorics* **34** (2013), 905–915.

[8] P. Erdős and T. Gallai, On maximal paths and circuits of graphs, *Acta Mathematica Academiae Scientiarum Hungaricae* **10** (1959), 337–356.

[9] R. Faudree and R. Schelp, Path Ramsey numbers in multicolorings, *Journal of Combinatorial Theory, Series B* **19** (1975), 150–160.

[10] H. Gebauer and F. Mousset, On Rainbow Cycles and Paths, arXiv:1207.0840 (2012).

[11] A. Gyárfás and M. Mhalla, Rainbow and orthogonal paths in factorizations of K_n, *Journal of Combinatorial Designs* **18** (2010), 167–176.

[12] P. Keevash, D. Mubayi, B. Sudakov and J. Verstraëte, Rainbow Turán problems, *Combinatorics, Probability and Computing* **16** (2007), 109–126.

[13] B. Lidický, H. Liu and C. Palmer, On the Turán number of forests, *Electronic Journal of Combinatorics* **20**(2) (2013), #P62.

[14] M. Maamoun and H. Meyniel, On a problem of G. Hahn about coloured Hamiltonian paths in K_{2^t}, *Discrete Mathematics* **51** (1984), 213–214.