Secrecy coverage in two dimensions

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***Abstract*—Imagine a sensor network consisting of base stations scattered at random over a large circular region R. Suppose that, in addition, R contains a smaller number of *eavesdroppers*, also scattered at random. Now suppose that each base station b can monitor a circular area whose radius is given by the distance from b to the nearest eavesdropper e. What is the probability that the entire region R can be monitored securely by the base stations? Extending work in [14], we estimate this probability, and thus provide detailed information on the maximum density of eavesdroppers that can be accommodated, while preserving complete coverage of R. From an engineering perspective, such coverage guarantees a scheme by which mobile stations can roam everywhere in R, and retain communication with a base station. From a mathematical perspective, the results reveal a new and surprising phenomenon, that the obstructions to coverage occur on a wide range of scales.**

I. Introduction

Imagine a sensor network consisting of base stations scattered at random over a large region R. Suppose that R also contains a smaller number of *eavesdroppers*, also scattered at random. Now suppose that each base station b can monitor a circular area whose radius is given by the distance from b to the nearest eavesdropper e. (Such a radius is justified by information-theoretic considerations – given b, e, and an intended mobile receiver m, b can choose a positive rate of transmission to m so that the *secrecy capacity* [9] is positive, providing that the distance between b and m is less than that between b and e.) We would like to know the maximum density of eavesdroppers that can be accommodated, while still guaranteeing that the entire region R can be monitored securely by the base stations. This would enable a scheme by which mobile stations could roam around everywhere in R, and always be reached securely by a base station; in such a scheme, the downlink is intrinsically secure, while the uplink, from the mobile to the base station, has to be secured by transmission of a one-time pad via the downlink.

In the case where the base stations are nodes of a communications network (rather than a sensor network), such considerations lead to the definitions of various types of *secrecy graph* [5], and the practical question of whether long-range communication is possible in the network translates to the mathematical question of whether *percolation* occurs in the graph. There has been much work on such questions, often using more elaborate physical layer models [13]. However, our question here has a somewhat different flavor, and concerns *coverage* rather than percolation.

We return to our original problem, which we model as follows. To eliminate boundary effects, we assume that the base stations and eavesdroppers form independent Poisson point processes in the plane. Specifically, let $\mathcal{P}$ and $\mathcal{P}'$ be independent Poisson processes, of intensities 1 and λ respectively, in $\mathbb{R}^2$. We will call the points of $\mathcal{P}$ *black points (base stations)* and the points of $\mathcal{P}'$ *red points (eavesdroppers)*. Place an open disc $D(p, r_p)$ of radius r_p around each black point $p \in \mathcal{P}$, where r_p is maximal so that $D(p, r_p) \cap \mathcal{P}' = \emptyset$. In other words, r_p, the distance from the black point p to the nearest red point $p' \in \mathcal{P}'$ to p, is the radius of the disc $D(p, r_p)$ that can be monitored by the base station at p. (The eavesdropper p' is almost surely unique.) We thus obtain a random set $\mathcal{A}_\lambda \subset \mathbb{R}^2$, the union of discs centered at the points of $\mathcal{P}$, which models the region that can be monitored by the base stations. Now let $B_n \subset \mathbb{R}^2$ be a fixed disc of area n, and set $\mathcal{A}_\lambda(B_n) = \mathcal{A}_\lambda \cap B_n$. Write $B_\lambda(n)$ for the event that $\mathcal{A}_\lambda(B_n)$ covers B_n (except for the points of $\mathcal{P}'$), and set $p_\lambda(n) = \mathbb{P}(B_\lambda(n))$, the probability of complete coverage. Our aim is to estimate $p_\lambda(n)$. Since adding red points makes coverage less likely, $p_\lambda(n)$ is a non-increasing function of λ, for fixed n. In addition, $p_\lambda(n)$ is non-increasing in n, with λ fixed, because increasing n corresponds to examining the random set A_λ over a larger area.

This model was defined and studied in [14] (see also [6]), where it was proved that if $\lambda^3 n \to \infty$ then $p_\lambda(n) \to 0$, while if $\lambda^3 n (\log n)^3 \to 0$ then $p_\lambda(n) \to 1$. In this paper, we prove that the correct indicator of coverage is $f(\lambda, n) = \lambda^3 n \log n$; if $f(\lambda, n) \to \infty$ then $p_\lambda(n) \to 0$, while if $f(\lambda, n) \to 0$ then $p_\lambda(n) \to 1$. Interestingly, the proofs indicate that there are obstructions on a range of scales; it seems that, close to the coverage threshold, there will be small uncovered regions whose widths range from around 1 to just above $n^{-1/6}$.

It is interesting to compare the results in this paper with those of the now classical *Gilbert model*. Here, we place discs of radius r in $\mathbb{R}^2$ so that their centers form a Poisson process of intensity 1, and again let $B_n \subset \mathbb{R}^2$ be a disc of area $n \gg r^2$. Once more, we ask for the probability that B_n is covered by the small discs. This question, inspired by biology [11], has a long history. Many detailed results are known about it [4], [7], [10], [8], [12]. For instance, writing

$$\pi r^2 = \log n + \log\log n + t,$$

Svante Janson proved in 1986 [8] that coverage occurs with probability asymptotically $e^{-e^{-t}}$, as $n \to \infty$. One approach to this result [2], [3] uses the fact that the obstructions to coverage are small uncovered regions, which essentially form their own Poisson process, of intensity e^{-t}/n. Although these uncovered regions may be of different shapes, they are all roughly the same size. In our variant of the problem, the disc radii are no longer independent, and there are many different obstructions of many different sizes.

Let us note that the problem of determining the *covered volume fraction* of $\mathcal{A}_\lambda$, which can be defined as $f_\lambda = \mathbb{P}(O \in \mathcal{A}_\lambda)$ (where O is the origin), was solved in [14]. The result is that

$$f_\lambda = 1 - \int_0^\infty f(t)e^{-t/\lambda}\,dt,$$

where $f(t)$ is the (currently unknown) probability density function for the volume of the cell containing the origin O in the Voronoi tessellation formed from $\mathcal{P} \cup \{O\}$, where $\mathcal{P}$ is a unit intensity Poisson process in $\mathbb{R}^2$. This is a genuinely different problem from the present one – it is entirely possible that the expected amount of uncovered area in B_n tends to zero, but that the probability that not all of B_n is covered tends to one. Indeed this does occur for certain values of the parameters.

As motivation for our main results, let us briefly state, and sketch the proof of, the result for the one-dimensional version of our problem. Here, we wish to cover an interval I_n of length n with small intervals centered at black points (a Poisson process with intensity 1), which in turn are stopped by red points (a Poisson process with intensity λ). Denoting the probability of such coverage by $p^1_\lambda(n)$, the result is as follows.

Theorem 1. *If $\lambda^2 n = x$, then $p^1_\lambda(n) \to e^{-4x}$.*

Proof. Let L be an interval of length ℓ between two consecutive red points in I_n. We wish to compute the probability that L is covered. With this in mind, let m be the midpoint of L, let x be the distance of the closest black point to m lying on the left of m, and let y be the distance of the closest black point to m lying on the right of m. Then coverage of L is determined solely by x and y. Indeed, coverage occurs if and only if $x + y < \ell/2$. Consequently, the probability that L is covered is just $\mathbb{P}(\mathrm{Po}(\ell/2) \geq 2) = 1 - e^{-\ell/2}(1 + \ell/2)$. Next, the unconditional probability that the interval between two consecutive red points is covered, obtained by integrating the above probability against the density function of ℓ, is $(1 + 2\lambda)^{-2} \sim 1 - 4\lambda$. Finally, since there are asymptotically $n\lambda \to \infty$ intervals between consecutive red points, and coverage fails independently in each one with probability asymptotically $4\lambda \to 0$, the number of failures is approximately Poisson with mean $4n\lambda^2 = 4x$, and the result follows. □

The above argument reveals that the obstructions to coverage will typically comprise two red points, distance $O(1)$ apart, without black points sufficiently close to their midpoint to ensure coverage of the interval between them. The set of such intervals is roughly four times as large as its subset consisting of consecutive red points with *no* black point between them. In other words, choosing λ to prohibit such pairs of consecutive red points provides a necessary condition for coverage, $\lambda^2 n \to 0$, which is in fact also sufficient, although such an argument gives the wrong constant in the exponent in Theorem 1. One might expect that a similar situation will exist in two dimensions, namely that if $\lambda^3 n = x$, then $p_\lambda(n)$ tends to e^{-cx}, or possibly some other function of x. The likely obstructions might be triples $\{p, q, r\}$ of red points forming a triangle T, whose sides and area are $O(1)$, and which contains no black points in its interior. However, as we shall show, the truth is more complicated.

II. Main results

Theorem 2. *If $\lambda^3 n \log n \to \infty$, then $p_\lambda(n) \to 0$.*

Proof. Our strategy will be to show that, under the hypothesis, the expected number of *good configurations* (defined below) tends to infinity. A routine application of the second moment method then shows that a good configuration occurs with high probability (probability tending to one). Finally, we show that a good configuration results in an uncovered region of B_n.

First, therefore, we define a good configuration. Such a configuration, illustrated (though not to scale) in Figure 1, consists of an ordered triple (p, q, r) of red points in B_n. p and q must lie at distance t, where $n^{-1/12} < t < 1$. r must lie at distance between $50/t$ and $100/t$ of p, in such a way that the angle rpq is between $\pi/4$ and $3\pi/4$. (The choice of these angles is somewhat arbitrary: all we need is that the angle rpq is bounded away from 0 and π.) Write ℓ_{ij} for the perpendicular bisector of ij, and S for the bi-infinite strip of width $||p-q||$ centered on ℓ_{pq}. For ease of explanation, suppose that the segment pq is horizontal, so that S is vertical, and that r lies above the line through p and q. ℓ_{pr} and ℓ_{qr} intersect the boundary ∂S of S in four points; suppose that the highest of these lies at height $h \leq 110/t$ above pq. Write $R \subset S$ for the rectangle with base pq and height $2h$ (containing all four intersections above), and $R' \subset S$ for its reflection in pq. A good configuration must also have no black points in the rectangular region $R \cup R'$. Note that the area of $R \cup R'$ is at most 440, so that, conditioned on the locations of p, q and r, the condition on the black points is satisfied with probability at least e^{-440}. Now, in a good configuration, given the position of p, q is constrained to lie in some annulus centered at p of area $2\pi t\,dt$, with $n^{-1/12} < t < 1$, and then r must lie in a region of area $7500\pi/4t^2$. Consequently, writing X for the number of good configurations, there exist absolute constants C and C' such that

$$\mathbb{E}(X) \geq C \int_{n^{-1/12}}^{1} \lambda n \cdot \lambda t^{-2} \cdot \lambda t\,dt = C'\lambda^3 n \log n \to \infty.$$

Second, we show that we can apply the second moment method to prove that, with high probability, $X \geq 1$. For this to work, we require an *upper* bound on λ; it will suffice to assume $\lambda^3 n \to 0$. Since $p_\lambda(n)$ is decreasing in λ, if we can prove that $p_\lambda(n) \to 0$ under the more restrictive hypotheses, the full result will follow. Tessellate B_n with squares of side length $n^{1/6}$, and color a square black if both of its “coordinates” are even. (Thus one out of every four squares

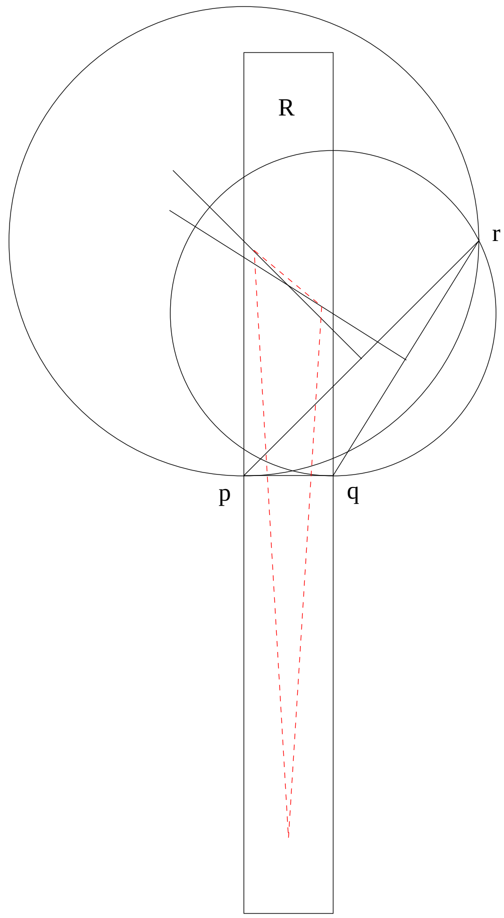


Fig. 1. A good configuration. r' and D_r are not shown, but D_r just fails to cover the curvilinear triangle with base pq. The dashed triangle is the Voronoi cell for the point s' slightly above the midpoint of pq.

is black.) We will only consider the black squares, which we label $S_1, S_2, \ldots, S_N$. Let the *apex* of a good configuration be the point furthest from the opposite side (r, in the above notation), and write X_i for the number of good configurations with apex in S_i. With high probability, each X_i will be either zero or one. Moreover, since the maximum diameter of a good configuration is $O(n^{1/12})$ by construction, the X_i are independent. Let $X' = \sum X_i$. Then $\mathbb{E}(X') \to \infty$ as above, and since

$$\mathbb{P}(X_i \geq 1) = O(\log n/n^{2/3}) \to 0,$$

it follows that

$$\mathrm{Var}(X') = \sum \mathrm{Var}(X_i) \sim \sum \mathbb{E}(X_i) = \mathbb{E}(X'),$$

and so by Chebyshev's inequality

$$\mathbb{P}(X = 0) \leq \mathbb{P}(X' = 0) \leq \frac{\mathrm{Var}(X')}{\mathbb{E}(X')^2} \sim \frac{1}{\mathbb{E}(X')} \to 0.$$

Finally, we explain why the presence of a good configuration prohibits full coverage. As above, suppose that pq is horizontal, and that r, and hence ℓ_{pr} and ℓ_{qr}, lie above pq. The idea is that part of ℓ_{pq} lying just above pq will be uncovered. Write m_0 for the midpoint of pq, and m_s for the point of ℓ_{pq} at height s above pq. Any black points lying in S and above pq are much closer to r than to p or q, and so their corresponding discs cannot cover m_0 or m_s, for $s \sim C/t$. Write q' for the intersection of ℓ_{pr} with ∂S lying above p, p' for the intersection of ℓ_{qr} with ∂S lying above q, and r' for the midpoint of the opposite side of R' from pq. The points p', q' and r' are the best locations to place black points for the purposes of covering points m_s, for small s. However, even their corresponding black discs fail to cover $m_{s'}$, for suitable s'. Specifically, write

$$\begin{aligned} D_p &= D(p', ||p' - q||) = D(p', ||p' - r||), \\ D_q &= D(q', ||q' - p||) = D(q', ||q' - r||), \\ D_r &= D(r', ||r' - p||) = D(r', ||r' - q||). \end{aligned}$$

If the distance of i' from pq is c_i/t, then the heights of D_p and D_q above m_0 are asymptotically $t^3/8c_i$, and D_r only covers m_s for $s < t^3/8c_r$ (asymptotically). However, by construction,

$$c_r \geq \tfrac{3}{2} \max\{c_p, c_q\},$$

so the point $m_{s'}$, for $s' = t^3/7c_r$, will be uncovered by $D_p \cup D_q \cup D_r$. Having identified s', it is straightforward to check that the Voronoi cell V of $\{s', p, q, r\}$ (shown dashed in Figure 1) is entirely contained in $R \cup R'$. Therefore, s' will not be covered by A_λ, since, by construction, $V \subset R \cup R'$ is free of black points; all black discs will be stopped by p, q, r, or another red point, before they cover s'. □

For $i = 0, 1, 2, \ldots$, let us say that a good configuration is of *type* i if the parameter $t = ||p-q||$ satisfies $2^{-(i+1)} \leq t < 2^{-i}$. Close to the threshold, for each i, there will be $O(\lambda^3 n) = o(1)$ good configurations of type i, so that, for fixed i, the probability that a good configuration of type i exists in B_n tends to zero. However, there are $C \log n$ possible types, and so, under the hypotheses of Theorem 2, *some* good configuration will occur in B_n with high probability. The next theorem shows that good configurations are essentially the only obstructions to coverage.

Theorem 3. *If* $\lambda^3 n \log n \to 0$, *then* $p_\lambda(n) \to 1$.

Proof. Suppose that $n \to \infty$ and also that $\lambda^3 n \log n \to 0$. First, we show that we need only worry about coverage of parts of B_n which are close (within distance $\sqrt{8 \log n}$) to a red point. To do this, we tessellate B_n with squares of side length $r = \sqrt{\log n}$. The probability that any small square of the tessellation contains no black point is $e^{-\log n} = n^{-1}$. Since there are $\sim n/\log n$ such squares, the expected number of them containing no black points is asymptotically $1/\log n \to 0$. Consequently, with high probability, every small square contains a black point. Now fix a small square S. If no point of S is within distance $\sqrt{2 \log n}$ of a red point, and if S contains a black point, then all of S will be covered by $\mathcal{A}_\lambda$. Therefore, with high probability, any point of B_n at distance more than $\sqrt{8\log n}$ from all red points will be covered by $\mathcal{A}_\lambda$, and we may assume this from now on.

It remains to show that the regions of B_n within distance $\sqrt{8\log n}$ from a red point are covered by A_λ. Color such regions yellow. In order to facilitate a division into cases, let us construct a graph $G = G(n, \mathcal{P}')$ on the red points by joining two red points if they lie within distance $R = R(n) = \sqrt{128\log n}$ of each other. (Such a graph is usually called a *random geometric graph*.) A routine calculation shows that, with high probability, the connected components of G consist of $o(n^{2/3}(\log n)^{-1/3})$ isolated vertices, $o(n^{1/3}(\log n)^{1/3})$ edges, $o(\log n)$ triangles, and $o(\log n)$ paths of length 2 (i.e., paths with 2 edges). We deal with each of these in turn; it will be convenient to consider a path of length 2 as a triangle, even though one of its edges is "long".

Isolated vertices. Consider the circles of radii $\sqrt{8\log n}$ and $\sqrt{32\log n}$ around each isolated red point, and divide the annulus between these circles into 6 equal "sectors", each of area $4\pi\log n$. With high probability, there is a black point inside each sector, and this black point is closer to the isolated vertex than to any other red point. But then the yellow region surrounding the isolated vertex is covered by A_λ.

Edges. For a fixed edge $e = pq \in E(G)$, where we may assume $p = (0,0)$ and $q = (t,0)$, consider the circles of radii $\sqrt{8\log n}$ and $\sqrt{32\log n}$ around p and q. Divide each half-annulus, between two concentric circles and lying outside the "critical strip" $S = [0,t] \times \mathbb{R}$, into 3 equal sectors, each of area $4\pi\log n$. With high probability, there is a black point inside each sector, and this black point is closer to p or q than to any other red point. Thus the yellow regions outside S are covered by A_λ. However, coverage of the yellow regions inside the critical strip S is not guaranteed. Indeed, the proof of Theorem 2 shows that such coverage is threatened by the presence of red points at distance $\sim C/t$ from e. G contains edges almost as short as $n^{-1/6}$, so such points may lie almost as far as $n^{1/6}$ from e, almost as much as the typical distance between red points.

We need to show that the edge $e = pq$ is, with high probability, covered from both above and below, so that the yellow regions inside S both above and below e are covered by A_λ. It will be sufficient to show that e is covered from above with high probability; an analogous argument will then deal with coverage from below. Let r be the closest point to p, under the condition that the angle rpq is between 0 and π (thus, in this case, r is "above" e), and write $s = ||r-p||$. With notation as in the proof of Theorem 2, the lines ℓ_{pr} and ℓ_{qr} intersect at height $h \geq \frac{s}{2\sqrt{3}}$ above e (see Figure 2). Now let ℓ be the line parallel to e, lying at height $\sqrt{2\log n}$ above e, and let T be the rectangle with base of length $\frac{t}{2}$ lying on ℓ, of height

$$\frac{h}{2} - \sqrt{2\log n} \geq \frac{s}{4\sqrt{3}} - \sqrt{2\log n} \geq \frac{s}{60},$$

and such that T is bisected by ℓ_{pq}. Every point of T lies below both ℓ_{pr} and ℓ_{qr}, and so is closer to p and q than r. Denoting the left and right halves of T by L and R respectively, we see

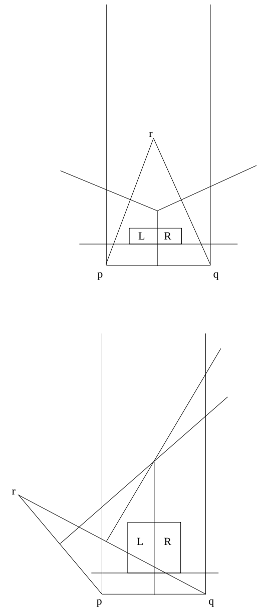


Fig. 2. Covering the edge pq from above

that if each of L and R contains a black point, then the entire yellow region inside S and above e will be covered by the discs centered at these two points. But, with probability at least $1 - 2e^{-st/240}$, L and R each do contain a black point. Therefore, there exist constants C and C' such that the expected number Y of edges not covered from above can be bounded by

$$\begin{aligned}
\mathbb{E}(Y) &\leq o(1) + C\lambda n \int_{n^{-1/6}}^{\sqrt{128\log n}} \lambda t \int_{\sqrt{128\log n}}^{\infty} \lambda s e^{-st/240}\, ds\, dt \\
&\leq o(1) + C\lambda n \int_{n^{-1/6}}^{\sqrt{128\log n}} \lambda t \int_{0}^{\infty} \lambda x t^{-2} e^{-x/240}\, dx\, dt \\
&= o(1) + C\lambda^3 n \int_{n^{-1/6}}^{\sqrt{128\log n}} t^{-1} \int_{0}^{\infty} x e^{-x/240}\, dx\, dt \\
&= o(1) + C'\lambda^3 n\log n \to 0.
\end{aligned}$$

Consequently, with high probability, the yellow regions close to all the edges in G are completely covered by A_λ.

Triangles. We expect $o(\log n)$ triangles T in G, and we will classify them by the length x of their smallest sides. In the first case, illustrated in the first two parts of Figure 3, no angle of T is greater than $\frac{9}{10}\pi$. Consider the disc D, centered at the circumcenter of T, of radius $\frac{x}{4}$. If each of the three sectors of D formed from the perpendicular bisectors of the sides of T contains a black point, then the entire interior of T is covered by A_λ; the exterior of T is easily seen to be covered with high probability. But each of these sectors has area at least $\frac{\pi}{20} \cdot \frac{x^2}{16} = \frac{\pi x^2}{320}$, so that the expected number T_1 of such triangles which are not entirely covered can be bounded by

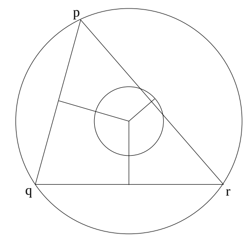
$$
\begin{aligned}
\mathbb{E}(T_1) &\leq o(1) + C\lambda^2 n \log n \int_{n^{-1/6}}^{\sqrt{128 \log n}} \lambda x e^{-\pi x^2/320}\, dx \\
&\leq o(1) + C\lambda^3 n \log n \int_0^{\infty} x e^{-\pi x^2/320}\, dx \\
&= o(1) + C'\lambda^3 n \log n \to 0,
\end{aligned}
$$

for some constants C and C'. In the second case, where one angle of T, say the angle at p, is greater than $\frac{9}{10}\pi$, we consider the two rectangles whose centers lie on ℓ_{pq} and ℓ_{pr}, halfway from pq (respectively pr) to the circumcenter of T, whose bases are parallel to the respective sides pq and pr, and whose heights and widths are $\frac{x}{10}$ and $\frac{x}{3}$ respectively (see the third part of Figure 3). If each half of each of these rectangles contains a black point, the interior of T is covered, and so the expected number T_2 of such triangles which are not entirely covered can be bounded by

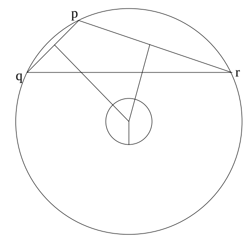
$$
\begin{aligned}
\mathbb{E}(T_2) &\leq o(1) + C\lambda^2 n \log n \int_{n^{-1/6}}^{\sqrt{128 \log n}} \lambda x e^{-\pi x^2/60}\, dx \\
&\leq o(1) + C\lambda^3 n \log n \int_0^{\infty} x e^{-\pi x^2/60}\, dx \\
&= o(1) + C'\lambda^3 n \log n \to 0,
\end{aligned}
$$

for some constants C and C'. Therefore, with high probability, the interiors of all the triangles in G are covered by A_λ, completing the proof of the theorem.

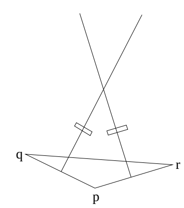
□



(a) Covering triangles: all angles acute



(b) Covering triangles: no large angle



(c) Covering triangles: one large angle

Fig. 3. Covering the interior of triangle pqr

III. HEURISTICS

I suspect that much more is true, and provable. Specifically, I now present a heuristic argument, which I hope to convert into a proof, suggesting that if $\lambda^3 n \log n = y$, then

$$
e^{-8\pi^2 y} \leq p_\lambda(n) \leq e^{-\frac{8}{3}\pi^2 y}.
$$

Suppose that $\lambda^3 n \log n = y$. We follow the strategy of the proof of Theorem 3, with a few modifications. Define the graph $G = G(n, \mathcal{P}')$ on the red points as in that proof. With high probability, the yellow regions associated with isolated vertices are still covered by black discs, even at this higher range of values of λ. I expect that more detailed estimates will show that the yellow regions inside and close to triangles in G are also still covered, again at this higher range of values of λ, and with high probability. Consequently, all the uncovered regions in B_n are associated with *edges* in G.

The detailed strategy for the remainder of the proof is as follows. First, we need to estimate the frequency of uncovered edges (i.e., edges in G whose associated yellow regions are uncovered by black discs). Suppose that this frequency is such that we expect cy uncovered edges in B_n. Then these uncovered edges can be well-approximated by a Poisson process in B_n (this can be verified using the Chen-Stein method [1]), and so the probability that there will be no uncovered edges will tend to e^{-cy}. But, following the above remarks, this is also the probability of coverage.

Unfortunately, estimating c itself seems quite hard, since, in contrast to the one-dimensional case, there is no simple necessary and sufficient condition for an edge of G to be covered by A_λ. The best we can do is describe a simple necessary condition for coverage (edges *not* satisfying this condition are termed *Type 2 edges*), and a corresponding simple sufficient condition for coverage (edges *not* satisfying such a condition are *Type 1 edges*). Type 2 edges provide a lower bound on c, and hence an upper bound on $p_\lambda(n)$, while Type 1 edges provide an upper bound on c, and a lower bound on $p_\lambda(n)$. To summarize, denoting the sets of Type 1, Type 2, and uncovered edges in B_n by T_1, T_2, and U, we have $T_2 \subset U \subset T_1$. We now turn to the precise descriptions of these types of edge.

Type 1 edges. With reference to Figures 1 and 2, let R be the rectangle whose base is parallel to pq and lies at height $\sqrt{2 \log n}$ above pq, whose top is parallel to pq and just touches the lowest of the four intersections of ℓ_{pr} and ℓ_{qr} with S, and whose sides are those of S itself. A sufficient condition for coverage of pq is that pq is *covered from above*, that is, there are sufficiently many black points in R to cover the yellow region above pq; the yellow region below pq is covered, for all such edges in B_n, with high probability. (This condition is not necessary, since black points below pq might by themselves cover the yellow regions on both sides.) We can estimate the number of Type 1 configurations by "projecting" the black points in R to the edge pq, resulting in a one-dimensional process on an interval of length 1 whose intensity is just the area of R, and applying (the proof of) Theorem 1. We will call a rectangle R whose black points do not cover pq from above a *blue rectangle*; Type 1 edges are those associated with blue rectangles.

Type 2 edges. Again with reference to Figures 1 and 2, let V be the dotted Voronoi cell corresponding to a point s', where s' has been chosen to maximize the area of V. A necessary condition for coverage of the yellow region above pq is coverage of the point s', and this occurs if and only if a black point lies in V. (This condition is clearly not sufficient.) A rough calculation shows that s', as defined in the proof of Theorem 2, is already almost optimal; the point on ℓ_{pq} which maximizes the area of V is at height asymptotically $t^2/8u$ above pq, where $||p-q||=t$, and where the circumcenter of pqr is at height u above pq. For this choice of s', V has area approximately $tu/2$, and so is free of black points with probability about $e^{-tu/2}$. Call a Voronoi cell V without black points a *green triangle*; Type 2 configurations are those associated with green triangles.

When estimating the frequencies of Type 1 and Type 2 edges, we may assume that $t \ll 1$, and indeed that $t \ll (\log n)^{-1/2}$, since even edges not satisfying the stronger restriction comprise an asymptotically negligible fraction of both types of edge. Also, the edges of both types are well-approximated by Poisson processes, so that if we expect $c_1 y$ edges of Type 1, and $c_2 y$ edges of Type 2, we will have $e^{-c_1 y} \le p_\lambda(n) \le e^{-c_2 y}$.

Suppose that the circumcenter of triangle pqr lies at height between u and $u+du$ above pq. This means that r must lie in an asymmetrical annulus of area $2\pi u\, du$. Under these circumstances, the rectangle R has area ut, and will be blue with probability $(1+ut/2)e^{-ut/2}$, while the Voronoi cell V has area $ut/2$, and will be green with probability $e^{-ut/2}$. Consequently, making the substitution $x=ut$ in both integrals,

$$\begin{aligned}
\mathbb{E}(|T_1|) &\sim \lambda n \int_{n^{-1/6}}^{\sqrt{128\log n}} 2\pi\lambda t \int_{\sqrt{128\log n}}^{\infty} \frac{2\pi\lambda u}{e^{ut/2}}\left(1+\frac{ut}{2}\right) du\, dt \\
&= 4\pi^2\lambda^3 n \int_{n^{-1/6}}^{\sqrt{128\log n}} \frac{1}{t}\, dt \int_{t\sqrt{128\log n}}^{\infty} x\left(1+\frac{x}{2}\right) e^{-x/2}\, dx \\
&\sim 4\pi^2\lambda^3 n \int_{n^{-1/6}}^{\sqrt{128\log n}} \frac{1}{t}\, dt \int_{0}^{\infty} x\left(1+\frac{x}{2}\right) e^{-x/2}\, dx \\
&\sim 8\pi^2\lambda^3 n\log n = 8\pi^2 y,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}(|T_2|) &\sim \lambda n \int_{n^{-1/6}}^{\sqrt{128\log n}} 2\pi\lambda t \int_{\sqrt{128\log n}}^{\infty} 2\pi\lambda u e^{-ut/2}\, du\, dt \\
&\sim 4\pi^2\lambda^3 n \int_{n^{-1/6}}^{\sqrt{128\log n}} \frac{1}{t}\, dt \int_{0}^{\infty} x e^{-x/2}\, dx \\
&\sim \frac{8}{3}\pi^2\lambda^3 n\log n = \frac{8}{3}\pi^2 y,
\end{aligned}$$

completing the argument.

The variable x in the above calculation can be interpreted as the amount by which a "generic" configuration has been "stretched"; the frequency of blue rectangles and green triangles corresponding to a fixed value of t and with $x \le ut \le x+dx$ is exponentially decreasing in x.

As explained above, it does seem likely that there exists a single constant c such that if $\lambda^3 n\log n = y$ then $p_\lambda(n) \to e^{-cy}$. It might even be possible to provide an explicit expression for c. Finally, it would be interesting to investigate the problem in higher dimensions.

IV. Conclusion

The main contribution of this paper has been to extend the work in [14] and provide detailed information on the probability of complete coverage in a sensor network populated by base stations and eavesdroppers. Close to the coverage threshold, the expected amount of uncovered area is very small, and perhaps negligible in practice. Nonetheless, the "unexpected" factor of $\log n$ in our results may be as high as 15 for some applications, which leads to a reduction in the threshold value of λ of about 2.5 (compared to the "naive" guess). It is surprising to me that such a simple mathematical problem seems to have such a complicated answer. I hope that the techniques developed in this paper will shed light on similar problems in the future.

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