Nagy's Conjecture on Extremal Densities

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Recall that the Turán number $\mathrm{ex}(n, F)$ of a graph F is the maximum number of edges in an F-free graph on n vertices.

Turán's theorem:

$$\mathrm{ex}(n, K_r) = \left(1 - \frac{1}{r-1} + o(1)\right)\frac{n^2}{2}$$

Erdős-Stone theorem:

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F)-1} + o(1)\right)\frac{n^2}{2}$$

where $\chi(F)$ is the chromatic number of F.

The extremal graphs F are Turán graphs, i.e., complete multipartite graphs.

Generalization: Write $N(G, H)$ for the number of copies of a (small, fixed, unlabelled) copy of G in a large graph H. If we fix the edge-density $e(H)/\binom{|H|}{2}$ of H, which graphs H minimize $N(G, H)$?

Hard

G bipartite: Sidorenko conjectured that H should be quasirandom.

$G = K_r$: Reiher proved that H should be close to a Turán graph.

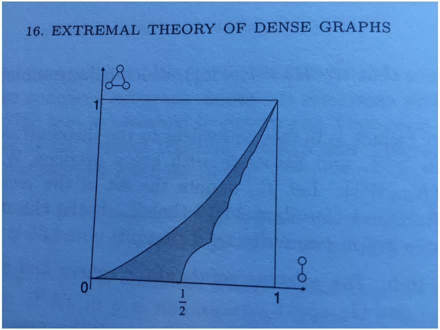
C. Reiher, The clique density theorem, *Annals of Mathematics* **184** (2016), 683–707.

$r = 3$ Razborov (flag algebras)

$r = 4$ Nikiforov

Instead: For fixed (small, unlabelled) G, and large H of fixed edge-density, how do we maximize $N(G, H)$?

Only solved for a few small graphs G, and two families of graphs.



Definition

$$\mathrm{ex}(n, e, G) = \max\{N(G, H) : |H| = n, e(H) \le e\}$$

Definition

Given n and $e \le \binom{n}{2}$, write $e = \binom{a}{2} + b$, where $0 \le b < a$. The quasi-clique K_n^e is a clique on a vertices, together with an additional vertex joined to b vertices of the clique, and $n-a-1$ isolated vertices. The quasi-star S_n^e is the complement of $K_n^{e'}$, where $e' = \binom{n}{2} - e$.

Theorem (Ahlswede and Katona 1978)

Let P_2 be the path with two edges. Then

$$\mathrm{ex}(n, e, P_2) = \max(N(P_2, S_n^e), N(P_2, K_n^e)).$$

The asymptotic maximizer is first a quasi-star and then a quasi-clique, with the switch occurring at edge-density $1/2$.

Theorem (Ahlswede and Katona 1978)

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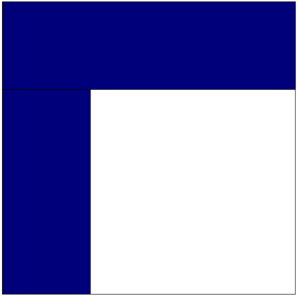
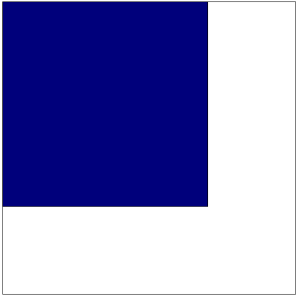


Quasi-star S_n^e



Quasi-clique K_n^e

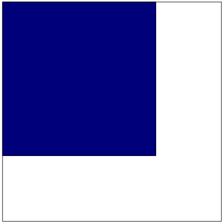
Theorem (Alon 1981)

Let G be a graph on v vertices. Then, as $n \to \infty$ with $\beta = 2e/n^2$ fixed,

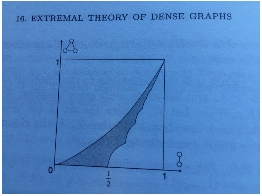
$$\mathrm{ex}(n, e, G) = N(G, K_n^e)(1 + o(1)) \Leftrightarrow \alpha^*(G) = v/2$$

where $\alpha^*(G)$ is the fractional independence number of G.

$\alpha^*(G) = v/2$ if and only if G has a spanning subgraph consisting of vertex-disjoint edges and cycles (e.g. paths with an odd number of edges, cycles, hamiltonian graphs).



Quasi-clique K_n^e



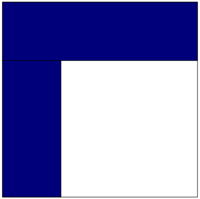
Profile for K_3

Theorem (Nagy 2017)

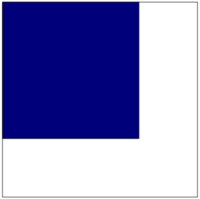
Let P_4 be the path with 4 edges. Then, as $n \to \infty$ with $\beta = 2e/n^2$ fixed,

$$\mathrm{ex}(n, e, P_4) = \max(N(P_4, S_n^e), N(P_4, K_n^e))(1 + o(1)).$$

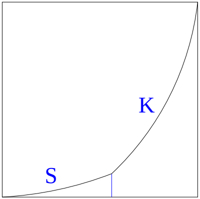
The asymptotic maximizer is first a quasi-star and then a quasi-clique, with the switch occurring at $\beta = 0.0865\ldots$.



Quasi-star S_n^e



Quasi-clique K_n^e



$N(n, e, P_4)$

Theorem (Nagy 2017)

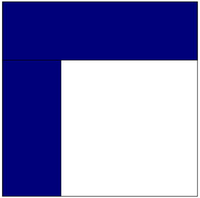
Let P_4 be the path with 4 edges. Then, as $n \to \infty$ with $\beta = 2e/n^2$ fixed,

$$\mathrm{ex}(n, e, P_4) = \max(N(P_4, S_n^e), N(P_4, K_n^e))(1 + o(1)).$$

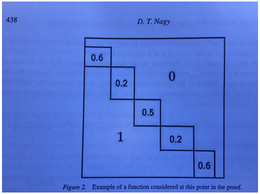
Definition

Let $A : [0,1)^2 \to [0,1]$ be integrable with $A(x,y) = A(y,x)$ and let

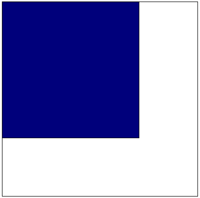
$$\ell(x) = \int_0^1 A(x,y)\,dy \quad \text{and} \quad S(A) = \int_0^1 \int_0^1 \ell(x)\ell(y) \min(\ell(x), \ell(y)\,dxdy.$$



Quasi-star S_n^e



From Nagy's paper



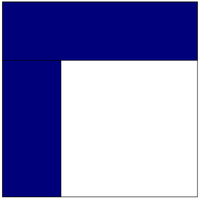
Quasi-clique K_n^e

Theorem (Reiher and Wagner 2018)

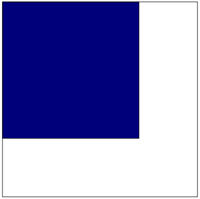
Let S_k be the star with k edges. Then, as $n \to \infty$ with $\beta = 2e/n^2$ fixed,

$$\mathrm{ex}(n, e, S_k) = \max(N(S_k, S_n^e), N(S_k, K_n^e))(1 + o(1)).$$

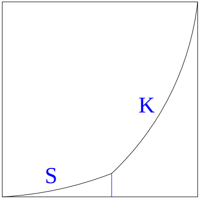
$2 \leq k \leq 30$: Kenyon, Radin, Ren and Sadun (2017)



Quasi-star S_n^e



Quasi-clique K_n^e



$N(n, e, S_k)$

Question (Nagy 2017)

Does there exist, for each graph G, a threshold $\beta_G < 1$ such that, for $\beta > \beta_G$, and as $n \to \infty$ with $\beta = 2e/n^2$ fixed,

$$\mathrm{ex}(n, e, G) = N(G, K_n^e)(1 + o(1)).$$

Yes (Gerbner, Nagy, Patkós, Vizer 2018+)

Theorem (Reiher and Wagner 2018)

Let S_k be the star with k edges. Then there exists $\beta_k < 1$ such that, for $\beta > \beta_k$, and as $n \to \infty$ with $\beta = 2e/n^2$ fixed,

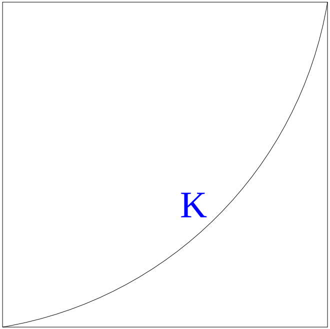
$$\mathrm{ex}(n, e, S_k) = N(S_k, K_n^e)(1 + o(1)).$$

Question (Nagy 2017)

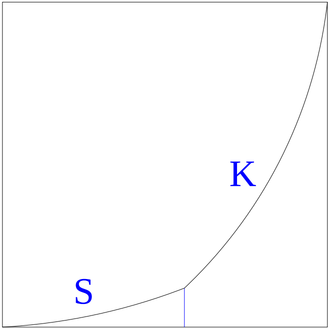
Is it true that, for each graph G,

$$\mathrm{ex}(n, e, G) = \max(N(S_k, S_n^e), N(S_k, K_n^e))(1 + o(1)).$$

More specifically, is every graph of type **K** or **SK**?



Type **K**



Type **SK**

No (Day and S. 2019+)

Definition

Given a graph G, a function $\phi : V(G) \to [0,1]$ such that $\phi(u) + \phi(w) \leq 1$ for all $uw \in E(G)$ is known as a fractional independence weighting of G.

Definition

The fractional independence number of G, written $\alpha^*(G)$, is defined as the maximum of $\sum_{u \in V(G)} \phi(u)$ over all fractional independence weightings of G.

Theorem (Nemhauser and Trotter 1974)

Every graph G has a maximal weighting (one that realizes $\alpha^*(G)$) in which all the weights are either $0, 1/2$ or 1.

Fix a labelling G_l of G; then $N_l(G_l, H) = N(G, H)|\mathrm{AutG}|$.

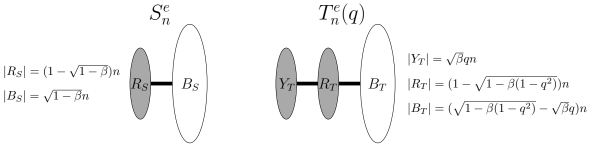
Definition

A graph homomorphism from G_l to H is a map $f : V(G_l) \to V(H)$ such that $f(u)f(w) \in E(H)$ for all $uw \in E(G_l)$. We write $\hom(G_l, H)$ for the number of homomorphisms from G_l to H.

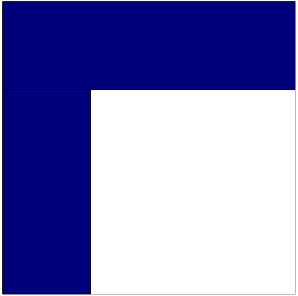
Definition

Given a family of graphs $(H_n)_{n \geqslant 1}$ such that $|V(H_n)| = n$, we define the homomorphism density $t(G_l, H_n)$ by the formula

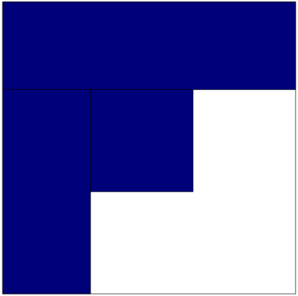
$$t(G_l, H_n) = \lim_{n\to\infty} \frac{N_l(G_l, H_n)}{n^v} = \lim_{n\to\infty} \frac{\hom(G_l, H_n)}{n^v}.$$



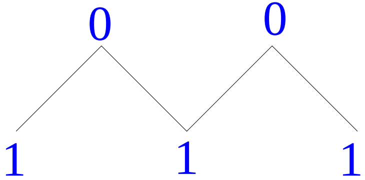
S_n^e and $T_n^e(q)$



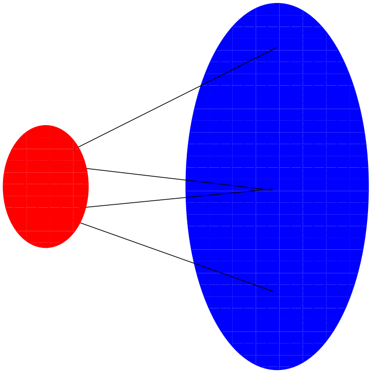
Quasi-star S_n^e



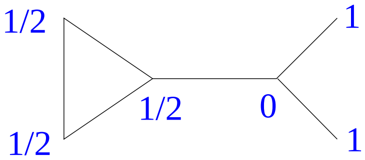
$T_n^e(q)$



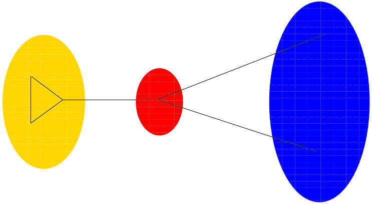
$\alpha(P_4) = 3$



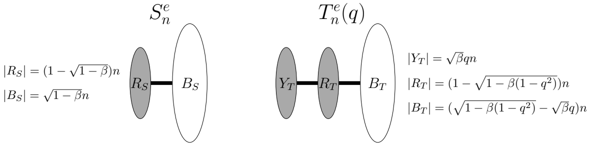
$t(P_4, S_n^e) = C\beta^2$



$\alpha^*(G_6) = 7/2$



$t(G_6, T_n^e(q)) = C\beta^{5/2}$



S_n^e and $T_n^e(q)$

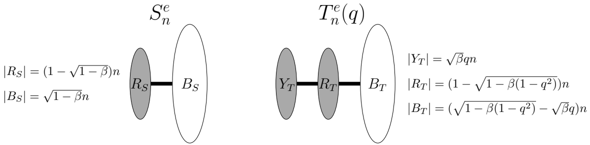
As $\beta \to 0$,

- $t(G_I, K_n^e)$ is determined by $v/2$
- $t(G_I, S_n^e)$ is determined by $\alpha(G)$
- $t(G_I, T_n^e(q))$ is determined by $\alpha^*(G)$

So, if $\alpha^*(G) > \max(v/2, \alpha(G))$, then

$$t(G_I, T_n^e(q)) >> \max(t(G_I, S_n^e), t(G_I, K_n^e))$$

as $\beta \to 0$, and thus Nagy's conjecture is false for G.



S_n^e and $T_n^e(q)$

Theorem (Day and S. 2019+)

Let G_l be a labelled graph on v vertices with no isolated vertices. Fix $q \in (0,1)$, and let $\beta \to 0$. Then there exist constants $C_1 = C_1(G_l, q) > 0$ and $C_2 = C_2(G_l) > 0$ such that the following all hold:

- $t\left(G_l, K_n^e\right) = \beta^{\frac{v}{2}}$
- $t\left(G_l, T_n^e(q)\right) = C_1\left(\beta^{v-\alpha^*(G)} + O\left(\beta^{v-\alpha^*(G)+\frac{1}{2}}\right)\right)$
- $t\left(G_l, S_n^e\right) = C_2\left(\beta^{v-\alpha(G)} + O\left(\beta^{v-\alpha(G)+1}\right)\right)$

Theorem (Day and S. 2019+)

Let G_l be a labelled graph on v vertices with no isolated vertices. Fix $q \in (0,1)$, and let $\beta \to 0$. Then there exist constants $C_1 = C_1(G_l, q) > 0$ and $C_2 = C_2(G_l) > 0$ such that the following all hold:

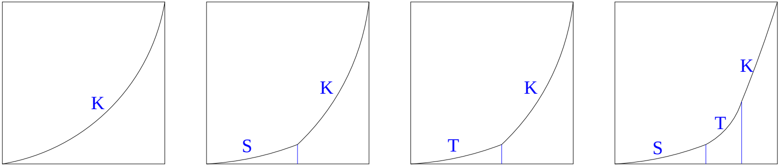
- $t\left(G_l, K_n^e\right) = \beta^{\frac{v}{2}}$
- $t\left(G_l, T_n^e(q)\right) = C_1\left(\beta^{v-\alpha^*(G)} + O\left(\beta^{v-\alpha^*(G)+\frac{1}{2}}\right)\right)$
- $t\left(G_l, S_n^e\right) = C_2\left(\beta^{v-\alpha(G)} + O\left(\beta^{v-\alpha(G)+1}\right)\right)$

Theorem (Janson, Oleszkiewicz, Ruciński 2004)

Let G be a graph on v vertices with fractional independence number $\alpha^*(G)$. Then, with $\beta = 2e/n^2$,

$$N(n,e,G) = \Theta(n^v \beta^{v-\alpha^*(G)}).$$

Lower bound: separate construction for each G
Upper bound: LP duality, Shearer's Lemma



Type **K** Type **SK** Type **TK** Type **STK**

Question 1

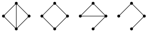
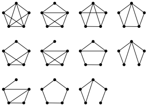
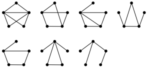
Is every graph of type **K**, **SK**, **TK** or **STK**?

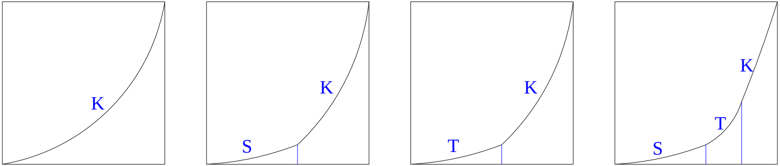
Question 2

Just comparing the families K_n^e, S_n^e and $T_n^e(q)$:
is every graph of type **K**, **SK**, **TK** or **STK**?

Question 3

Is G_6 of type **TK**?

Connected graphs on 3 vertices	$\alpha(G)$	$\alpha^*(G)$	Type	Proof
	1	$\frac{3}{2}$	**K**	• Alon, (Kruskal, Katona)
	2	2	**SK**	• Ahlswede, Katona
Connected graphs on 4 vertices				
	1	2	**K**	• Alon, (Kruskal, Katona)
	2	2	**K**	• Alon
	3	3	**SK**	• Kenyon, Radin, Ren, Sadun • Reiher, Wagner
Connected graphs on 5 vertices				
	1	$\frac{5}{2}$	**K**	• Alon, (Kruskal, Katona)
	2	$\frac{5}{2}$	**K**	• Alon
	3	3	**SK or STK**	• Lemma 2 + Corollary 3 (assuming Conjectures 1 and 2)
	3	3	**SK**	• Nagy
	4	4	**SK**	• Kenyon, Radin, Ren, Sadun • Reiher, Wagner



Type **K** Type **SK** Type **TK** Type **STK**

Question 4

Are there any graphs of type **STK**?

Question 5

For each G, is $q_{\max}(G, \beta)$ is an increasing function of β?

Question 6

As $\beta \to 0$, is every graph of type **S**, **T** or **K**?

Thank you for your attention!