Dirac's Theorem

Theorem 1. *Every graph G with $n \geq 3$ vertices and minimum degree $\delta(G) \geq n/2$ has a Hamilton cycle.*

Proof. Suppose that $G = (V, E)$ satisfies the hypotheses of the theorem. Then G is connected, since otherwise the degree of any vertex in a smallest component C of G would be at most $|C| - 1 < n/2$, contradicting the hypothesis $\delta(G) \geq n/2$.

Let $P = x_0 x_1 \cdots x_k$ be a longest path in G. Since P cannot be extended to a longer path, all the neighbours of x_0 and all the neighbours of x_k lie on P. Hence, at least $n/2$ of the vertices $x_0, \ldots, x_{k-1}$ are adjacent to x_k, and at least $n/2$ of the vertices $x_1, \ldots, x_k$ are adjacent to x_0. Another way of saying the second part of the last sentence is: at least $n/2$ of the vertices $x_i \in \{x_0, \ldots, x_{k-1}\}$ are such that $x_0 x_{i+1} \in E$. Combining both statements and using the pigeonhole principle, we see that there is some x_i with $0 \leq i \leq k-1, x_i x_k \in E$ and $x_0 x_{i+1} \in E$ (see the figure below).

We claim that the cycle

$$C = x_0 x_{i+1} x_{i+2} \cdots x_{k-1} x_k x_i x_{i-1} \cdots x_1 x_0 = x_0 x_{i+1} P x_k x_i P x_0$$

is a Hamilton cycle of G. Otherwise, since G is connected, there would be some vertex x_j of C adjacent to a vertex y not in C, so that $e = x_j y \in E$. But then we could attach e to a path ending in x_j containing k edges of C, constructing a path in G longer than P. □