

GELATION IN VECTOR MULTIPLICATIVE COALESCENCE AND EXTINCTION IN MULTI-TYPE POISSON BRANCHING PROCESSES

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ABSTRACT. Random coalescent processes and branching processes are two fundamental constructs in the field of stochastic processes, each with a rich history and a wide range of applications. Though developed within distinct contexts, in this note we present a novel connection between a multi-type (vector) multiplicative coalescent process and a multi-type branching process with Poisson offspring distributions. More specifically, we show that the equations that govern the phenomenon of gelation in the vector multiplicative coalescent process are equivalent to the set of equations that yield the extinction probabilities of the corresponding multi-type Poisson branching process. We then leverage this connection with two applications, one in each direction. The first is a new quick proof of gelation in the vector multiplicative coalescent process using a well known result of branching processes, and the second is a new series expression for the extinction probabilities of the multi-type Poisson branching process using results derived from the theory of vector multiplicative coalescence. While the correspondence is fairly straightforward, it illuminates a deep connection between these two paradigms which we hope will continue to reveal new insights and potential for cross-disciplinary research.

1. VECTOR MULTIPLICATIVE COALESCENT PROCESSES

In this section, we give an overview of the theory of vector multiplicative coalescent processes. The complete theory was derived by two of the authors in [13]. Following the convention in [13], to define the vector multiplicative coalescent process, we use bra-ket notation. Specifically, $|\mathbf{x}\rangle$ will denote the column vector representation of vector $\mathbf{x} \in \mathbb{R}^k$, and $\langle \mathbf{x}|$ will denote the row vector representation of vector $\mathbf{x} \in \mathbb{R}^k$. For $c \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^k$, $c|\mathbf{x}\rangle$ will represent the product $c\mathbf{x}$, a column vector. Correspondingly, $\langle \mathbf{x}|\mathbf{y}\rangle = \langle \mathbf{y}|\mathbf{x}\rangle$ will be the dot product of \mathbf{x} and \mathbf{y} in \mathbb{R}^k . Finally, for a matrix $M \in \mathbb{R}^{k \times k}$, $\langle \mathbf{x}|M|\mathbf{y}\rangle$ will represent the product $\mathbf{x}^\top M \mathbf{y}$ resulting in a scalar.

Consider a system with k types of particles: $1, \dots, k$, or equivalently, a system where all the particles are partitioned into k parts. For a given vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_k) \in (0, \infty)^k$, the process begins with $\langle \boldsymbol{\alpha} n | \mathbf{1} \rangle$ singletons distributed between the k types so that for each i , there are $\alpha_i n$ particles of type i .

Let $V \in \mathbb{R}^{k \times k}$ be a nonnegative, irreducible and symmetric matrix, which we will refer to as the **partition interaction matrix**. A particle of type i bonds with a particle of type j with the intensity rate $v_{i,j}/n$, where $v_{i,j}$ is the (i, j) element in the matrix V . The bonds are formed independently. This process is called *vector-multiplicative coalescent*. Note that the irreducible condition on V ensures that particles of each type could ultimately coalesce with particles of every other type at some time in the vector-multiplicative coalescent process.

The *size* of a cluster of particles is represented by a k -dimensional vector $\mathbf{x} = (x_1, x_2, \dots, x_k) \in \mathbb{Z}_+^k$ such that $\sum_{i=1}^k x_i > 0$, where each x_i represents the number of particles of type i (or in part i) in the cluster. Each cluster of weight \mathbf{x} bonds together x_1, x_2, \dots, x_k particles of corresponding types $1, 2, \dots, k$.

Key words and phrases. Vector-multiplicative coalescent, Gelation, Multitype branching processes, Extinction, Lambert-Euler inversion.

At any given time t , each pair of clusters with respective size vectors \mathbf{x} and \mathbf{y} will coalesce into a cluster of size $\mathbf{x} + \mathbf{y}$ with rate $K(\mathbf{x}, \mathbf{y})/n$, where

$$(1) \quad K(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x} | V | \mathbf{y} \rangle.$$

The last merger will create a cluster of size αn . Clusters coalesce independently of other clusters.

The kernel $K(\mathbf{x}, \mathbf{y})$ defined in (1) will be referred to as the *vector-multiplicative kernel*. The kernel is symmetric

$$K(\mathbf{x}, \mathbf{y}) = K(\mathbf{y}, \mathbf{x}) \quad \text{for all vectors } \mathbf{x}, \mathbf{y}$$

and bilinear

$$K(c_1 \mathbf{x} + c_2 \mathbf{y}, \mathbf{z}) = c_1 K(\mathbf{x}, \mathbf{z}) + c_2 K(\mathbf{y}, \mathbf{z}) \quad \text{for all vectors } \mathbf{x}, \mathbf{y}, \mathbf{z} \text{ and scalars } c_1, c_2.$$

In the vector-multiplicative coalescent process, let $\zeta_{\mathbf{x}}^{[n]}(t)$ denote the number of clusters of size \mathbf{x} at time $t \geq 0$. The initial values are $\zeta_{\mathbf{x}}^{[n]}(0) = \alpha_i n$ if $\mathbf{x} = \mathbf{e}_i$, where \mathbf{e}_i is the i -th standard basis vector and $\zeta_{\mathbf{x}}^{[n]}(0) = 0$ otherwise. The process

$$\mathbf{ML}_n(t) = \left(\zeta_{\mathbf{x}}^{[n]}(t) \right)_{\mathbf{x} \in \mathbb{Z}_+^k : \langle \mathbf{x} | \mathbf{1} \rangle > 0}$$

that counts clusters of all sizes in the vector-multiplicative coalescent process is the corresponding **Marcus-Lushnikov process**. We define $\zeta_{\mathbf{x}}(t) = \lim_{n \rightarrow \infty} \frac{\zeta_{\mathbf{x}}^{[n]}(t)}{n}$ to yield the deterministic limiting fraction of clusters of each size \mathbf{x} . The differential equations that govern the dynamics of $\zeta_{\mathbf{x}}(t)$ are called the *Smoluchowski coagulation system of equations* and are given by

$$(2) \quad \frac{d}{dt} \zeta_{\mathbf{x}}(t) = -\zeta_{\mathbf{x}} \sum_{\mathbf{y}} \zeta_{\mathbf{y}} \langle \mathbf{x} | V | \mathbf{y} \rangle + \frac{1}{2} \sum_{\mathbf{y}, \mathbf{z} : \mathbf{y} + \mathbf{z} = \mathbf{x}} \langle \mathbf{y} | V | \mathbf{z} \rangle \zeta_{\mathbf{y}} \zeta_{\mathbf{z}}$$

with initial conditions $\zeta_{\mathbf{x}}(0) = \alpha_i$ if $\mathbf{x} = \mathbf{e}_i$, and $\zeta_{\mathbf{x}}(0) = 0$ otherwise. These equations describe the evolution of the size distribution of clusters in a system where clusters merge multiplicatively as defined in (1). More specifically, the first term on the right hand side in (2) corresponds to the depletion of clusters of size \mathbf{x} after merging with other clusters and the second term corresponds to coagulation of smaller clusters to produce one of size \mathbf{x} . For further details on Smoluchowski equations in various settings including applications, we refer the reader to [17, 15, 4, 12, 6].

In [13], it was shown that the solutions to the Smoluchowski equations and the following **modified Smoluchowski equations (MSE)** given by

$$(3) \quad \frac{d}{dt} \zeta_{\mathbf{x}}(t) = -\zeta_{\mathbf{x}} \langle \mathbf{x} | V | \alpha \rangle + \frac{1}{2} \sum_{\mathbf{y}, \mathbf{z} : \mathbf{y} + \mathbf{z} = \mathbf{x}} \langle \mathbf{y} | V | \mathbf{z} \rangle \zeta_{\mathbf{y}} \zeta_{\mathbf{z}}$$

with the same initial conditions coincide as long as the second order moments are convergent. The unique solution $\zeta_{\mathbf{x}}(t)$ of the above modified Smoluchowski equations was derived in [13]. Specifically, for a vector $\mathbf{x} \in \mathbb{Z}_+^k$ let $\mathbf{x}! = x_1! x_2! \dots x_k!$ and for vectors \mathbf{a} and \mathbf{b} in \mathbb{R}^k let $\mathbf{a}^{\mathbf{b}} = a_1^{b_1} a_2^{b_2} \dots a_k^{b_k}$ whenever $a_i^{b_i}$ is uniquely defined for all i . Now, consider a complete graph K_k consisting of vertices $\{1, \dots, k\}$ with weights $w_{i,j} = w_{j,i} \geq 0$ assigned to its edges $[i, j]$ ($i \neq j$). Let the weight $W(\mathcal{T})$ of a spanning tree \mathcal{T} be the product of the weights of all of its edges. Finally, let $\tau(K_k, w_{i,j}) = \sum_{\mathcal{T}} W(\mathcal{T})$ denote the *weighted spanning tree enumerator*, i.e., the sum of weights of all spanning trees in K_k . In terms of these notations, the solution to the modified Smoluchowski equations that yield the limiting fraction of clusters of size \mathbf{x} is given by

$$(4) \quad \zeta_{\mathbf{x}}(t) = \frac{1}{\mathbf{x}!} \alpha^{\mathbf{x}} \frac{\tau(K_k, x_i x_j w_{i,j})}{\mathbf{x}^{\mathbf{1}}} (V \mathbf{x})^{\mathbf{x}-\mathbf{1}} e^{-\langle \mathbf{x} | V | \alpha \rangle t} t^{\langle \mathbf{x} | \mathbf{1} \rangle - 1}.$$

1.1. Random graphs and an alternative proof of equation (4). The following alternative proof of (4) is of independent interest, as it uses the connection between the vector multiplicative coalescent process C and the theory of random graphs. This connection works as follows: clusters of particles in C correspond to small connected components of a certain random graph G . The so-called *giant component* of G corresponds to the *gel* in C - for details, see the next section. The correspondence between C and G was first described in [13], to which we refer the reader for a full account. Here, we include only enough detail to describe the proof of (4).

The random graph G models the coalescent process at a fixed time t . The n particles in C correspond to the n vertices of G , and, as before, we will let $n \rightarrow \infty$. The numbers $\alpha_i n$ of particles of each type in C correspond to the sizes of the parts A_i of G , so that G consists of (almost exactly) n vertices, divided into k parts A_1, \dots, A_k , in such a way that part A_i contains $\alpha_i n$ vertices. Each potential edge $x_i x_j$ of G (with $x_i \in A_i$ and $x_j \in A_j$) is included in $E(G)$, the edge set of G , with probability

$$1 - e^{-v_{ij}t/n} = \frac{v_{ij}t}{n} + O\left(\left(\frac{v_{ij}t}{n}\right)^2\right).$$

Here, $V = (v_{ij})$ is the partition interaction strength matrix defined above. The exponential formulation is necessary so that G can be seen as a snapshot of a certain memoryless process. Finally, for our purposes, t will be a constant, not depending on n . Note that if $k = \alpha_1 = v_{11} = 1$ then \mathbf{G} is (modulo the approximation above) just the *Erdős-Rényi graph* $G(n, t/n)$, with average degree (approximately) t . (Also note that we are allowing edges inside the parts, so that our graphs are not necessarily k -partite.)

With α and V fixed, there will be a critical value t' of t below which there is almost surely no giant component, and above which there is almost surely a giant component. Writing $A = \text{diag}(\alpha)$, this critical value t' is such that the largest eigenvalue of the matrix $t'VA$ equals 1 (i.e., t' is the reciprocal of the spectral radius of VA).

Next, again with V and α fixed, and for given values of $\mathbf{x} = (x_1, \dots, x_k)$ and t , we seek the scaled number of *finite* components in \mathbf{G} containing x_i vertices of type i . By “scaled”, we mean the number of such components divided by n . Call (the limit as $n \rightarrow \infty$ of) this scaled quantity $\phi_{\mathbf{x}}(t)$. Note that the process undergoes a qualitative change when $t = t'$, but, for a given \mathbf{x} , the functions $\phi_{\mathbf{x}}(t)$ are in fact continuous (and probably even differentiable) for all t .

The following theorem, connecting the process C and the graph G , appears in [13] (see also [1]).

Theorem 1.1. [1] *For given α, V and for any \mathbf{x} and $t < t' = T_{\text{gel}}$, we have*

$$\phi_{\mathbf{x}}(t) = \zeta_{\mathbf{x}}(t),$$

where the functions $\zeta_{\mathbf{x}}(t)$ are the solutions of the modified Smoluchowski equations (3) above.

Consequently, at least in the pre-gelation interval, we can use the random graph formulation to compute $\zeta_{\mathbf{x}}(t)$. We choose a method described on page 188 in Chapter 11 of [2].

For simplicity, we first address the case where $k = \alpha_1 = v_{11} = 1$. In this case, setting $\mathbf{x} = k$, we have

$$(5) \quad \zeta_k(t) = \phi_k(t) = \frac{k^{k-2} t^{k-1} e^{-kt}}{k!}$$

The random graphs proof of this is as follows. We calculate $p_k(t) = k\phi_k(t)$, which is the probability that a given vertex lies in a connected component G_k of size k (which, with probability tending to 1, is a tree - see [5] for a proof). Now, setting $p = t/n$ for the edge probability, we see that:

- There are $\binom{n-1}{k-1}$ choices for the other vertices in G_k
- There are k^{k-2} labeled trees on these chosen vertices
- Each tree is present with probability p^{k-1}

- The edges between G_k and the other vertices are missing with probability $(1-p)^{k(n-k)}$

Therefore, as $n \rightarrow \infty$,

$$\begin{aligned} p_k(t) &\sim \binom{n-1}{k-1} \cdot k^{k-2} \cdot p^{k-1} \cdot (1-p)^{k(n-k)} \sim \frac{n^{k-1}}{(k-1)!} \cdot k^{k-2} \cdot \left(\frac{t}{n}\right)^{k-1} \cdot e^{-pkn} \\ &= \frac{1}{(k-1)!} \cdot k^{k-2} \cdot t^{k-1} \cdot e^{-tk} = \frac{k^{k-1} t^{k-1} e^{-kt}}{k!}, \end{aligned}$$

which, recalling that $p_k(t) = k\phi_k(t)$, establishes (5).

Next we consider the general case. For this we once again require the *weighted spanning tree enumerator* defined above, although here it arises in a slightly different form. Write $K_{\mathbf{x}}(V)$ for the complete graph on $|\mathbf{x}| = \sum_i x_i$ vertices with x_i vertices of type i , and where the weight of an edge between vertices of types i and j is v_{ij} . Suppressing the dependence on V , write $T_{\mathbf{x}}$ for the sum of the weights of all the spanning trees of $K_{\mathbf{x}}(V)$. Then $T_{\mathbf{x}} = \tau(K_k, x_i x_j v_{ij})$, so that the two definitions of the weighted spanning tree enumerator agree.

Now, generalizing the argument above, pick a random vertex v . The probability that v is of type i is α_i . For simplicity of notation, suppose that $i = 1$. We aim to compute $p_{\mathbf{x}}(t)$, which is the probability that v lies in a connected component $G_{\mathbf{x}}$ of type \mathbf{x} . Again, with probability tending to 1, $G_{\mathbf{x}}$ is a tree [5]. Omitting floor and ceiling functions:

- There are $\binom{\alpha_1 n}{x_1-1} \binom{\alpha_2 n}{x_2} \cdots \binom{\alpha_k n}{x_k}$ choices for the other vertices in $G_{\mathbf{x}}$
- These form a tree with probability $T_{\mathbf{x}} \cdot (t/n)^{|\mathbf{x}|-1}$
- The edges from $G_{\mathbf{x}}$ to $G \setminus G_{\mathbf{x}}$ are missing with probability $\sim \exp(-\sum_i \sum_j x_i v_{ij} \alpha_j t)$

Therefore, as $n \rightarrow \infty$,

$$\begin{aligned} p_{\mathbf{x}}(t) &\sim \sum_i x_i \frac{n^{|\mathbf{x}|-1} \alpha_1^{x_1} \cdots \alpha_k^{x_k}}{x_1! \cdots x_k!} \cdot T_{\mathbf{x}} \cdot \left(\frac{t}{n}\right)^{|\mathbf{x}|-1} \cdot \exp\left(-\sum_i \sum_j x_i v_{ij} \alpha_j t\right) \\ &= \sum_i x_i \frac{\alpha_1^{x_1} \cdots \alpha_k^{x_k}}{x_1! \cdots x_k!} \cdot T_{\mathbf{x}} \cdot t^{|\mathbf{x}|-1} \cdot \exp\left(-\sum_i \sum_j x_i v_{ij} \alpha_j t\right), \end{aligned}$$

so that, since

$$p_{\mathbf{x}}(t) = \phi_{\mathbf{x}}(t) \sum_i x_i,$$

we have

$$\zeta_{\mathbf{x}}(t) = \phi_{\mathbf{x}}(t) = \frac{\alpha_1^{x_1} \cdots \alpha_k^{x_k}}{x_1! \cdots x_k!} \cdot T_{\mathbf{x}} \cdot t^{|\mathbf{x}|-1} \cdot \exp\left(-\sum_i \sum_j x_i v_{ij} \alpha_j t\right),$$

proving (4).

1.2. Gelation and Lambert-Euler Inversion. It is natural to ask if the total mass for a system described by the Smoluchowski coagulation model is conserved. In [11, 16, 18, 19], it was shown that for certain classes of coagulation kernels, this is not always possible as a system can undergo a phase transition. For example, Hendriks et al. [9] studied this phase transition for the model with the multiplicative kernel (in one-dimensional setting), and they showed that the total mass decreases after a finite time, observing a surprising phenomenon known as *gelation*. Broadly speaking, gelation is the break down of conservation of mass in the system and can be physically interpreted as being caused by the formation of a giant cluster called the *gel*.

For the coalescent process defined in section (1), the initial total mass vector is assumed to be $\sum_{\mathbf{x}} \zeta_{\mathbf{x}}(0)|\mathbf{x}\rangle = |\boldsymbol{\alpha}\rangle$. Therefore, we define the gelation as the loss of total mass $\sum_{\mathbf{x}} \zeta_{\mathbf{x}}(t)|\mathbf{x}\rangle$ of the system after a critical time T_{gel} called *gelation time*. That is

$$(6) \quad T_{gel} = \inf \left\{ t > 0 : \sum_{\mathbf{x}} \zeta_{\mathbf{x}}(t)|\mathbf{x}\rangle < |\boldsymbol{\alpha}\rangle \right\}.$$

In [13], the existence of gelation was proved for the vector multiplicative coalescent process and the gelation time was given by

$$(7) \quad T_{gel} = \frac{1}{\rho(VD[\alpha_i])},$$

where for a vector $\mathbf{x} \in \mathbb{R}^k$ with coordinates x_i , $D[x_i]$ denotes the diagonal matrix with entries x_i , and $\rho(A)$ denotes the spectral radius of matrix A .

The existence of gelation for the vector multiplicative coalescent process was proved by solving a multidimensional Lambert-Euler inversion problem. This is a higher dimensional generalization of the equation originally studied by Lambert and followed up by Euler that gave rise to the well known Lambert W function [14, 8]. For a given vector $\boldsymbol{\alpha} \in (0, \infty)^k$, consider the regions

$$(8) \quad R_0 = \{ \boldsymbol{\alpha} \in (0, \infty)^k : \rho(VD[\alpha_i]) < 1 \}$$

its closure within $(0, \infty)^k$,

$$(9) \quad \bar{R}_0 = \{ \boldsymbol{\alpha} \in (0, \infty)^k : \rho(VD[\alpha_i]) \leq 1 \},$$

and the complement of \bar{R}_0 within $(0, \infty)^k$,

$$(10) \quad R_1 = \{ \boldsymbol{\alpha} \in (0, \infty)^k : \rho(VD[\alpha_i]) > 1 \}.$$

The following theorem was proved in [13].

Theorem 1.2 (Multidimensional Lambert-Euler inversion). *Consider a nonnegative irreducible symmetric matrix $V \in \mathbb{R}^{k \times k}$. Then, for any given $\boldsymbol{\alpha} \in (0, \infty)^k$ and $t \geq 0$, there exists a unique vector $\mathbf{y} \in \bar{R}_0$ such that*

$$(11) \quad y_i e^{-(\mathbf{e}_i | V | \mathbf{y})} = \alpha_i t e^{-(\mathbf{e}_i | V | \boldsymbol{\alpha} t)} \quad i = 1, \dots, k.$$

Moreover, if $\boldsymbol{\alpha} t \in \bar{R}_0$, then $\mathbf{y} = \boldsymbol{\alpha} t$. If $\boldsymbol{\alpha} t \in R_1$, then $\mathbf{y} < \boldsymbol{\alpha} t$ ($y_i < \alpha_i t$ for all i), i.e., \mathbf{y} is the smallest solution of (11).

Finally, in [13], it was shown that the total mass of the vector multiplicative coalescent process could be expressed in terms of the solution of the multidimensional Lambert-Euler inversion problem as the following:

$$(12) \quad \sum_{\mathbf{x}} \zeta_{\mathbf{x}}(t)|\mathbf{x}\rangle = \frac{1}{t} |\mathbf{y}(t)\rangle$$

Then Theorem 1.2 proves the existence of the gelation phenomenon for the vector multiplicative coalescent process and the gelation time given in (7).

Remark 1.3. *The one dimensional Lambert-Euler inversion reads as*

$$x(t) = \min \{ x \geq 0 : x e^{-x} = t e^{-t} \}.$$

In their now famous paper [7], Erdős and Rényi used the function $x(t)$ to establish the formation of a giant component in the theory of random graphs. In fact, in random graph setting, the quantity $x(t)$ can be interpreted as the average degree outside of the giant component in an Erdős-Rényi graph. It would be interesting to see if (11) can be used to obtain refined information about the proportion of vertices in the giant component of a random multipartite graphs, in the same spirit as the classical case.

Note that in this one-dimensional setting, $x(t) = t$ on $(0, 1]$, and $x(t) < t$ on $(1, \infty)$. Moreover, $x(t) \downarrow 0$ monotonically as $t \rightarrow \infty$. On the other hand, the post-gelation behavior of the multi-dimensional solution to (11) can differ from the one-dimensional case. For example, consider the bipartite case with $V = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ and $\alpha = (15, 2)$. Then, the critical time is given by $T_{gel} = \frac{1}{\sqrt{30}}$, and for $t > T_{gel}$, the graph of $y(t) = (y_1(t), y_2(t))$ is as follows.

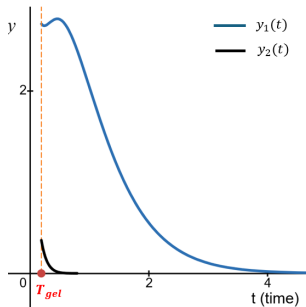


FIGURE 1. Post-gelation behavior of Lambert-Euler inversion

Observe that the function $y_1(t)$ decreases, increases, and finally decreases as t increases. Numerical simulation seems to imply that there is a critical value for the ratio of the parameters in α after which the solutions $(y_1(t), y_2(t))$ are no longer monotonically decreasing.

2. MULTI-TYPE POISSON BRANCHING PROCESSES

In this and the next sections, we give an overview of the relevant theory of multi-type branching processes which is a generalization of the classical Galton-Watson process, extending it to multiple types of individuals. A full theory can be found in [3]. This multi-type model is useful in various fields, including biology, physics, and epidemiology, to study populations with different types of individuals and how these types evolve over time. Recall that in the classical Galton-Watson process, there is only one type of individual, whereas in the multi-type case, there are k different types: $1, 2, \dots, k$. We shall assume that the process starts with only one individual of a specific type initially, i.e. $\mathbf{Z}_0 = \mathbf{e}_i$. Let $\mathbf{Z}_n = (Z_{1n}, Z_{2n}, \dots, Z_{kn})$ be the k -dimensional vector representing the number of individuals of each type in the n th generation of the branching process. The number of each type are independent random variables. Let $\mathbf{z} = (z_1, \dots, z_k)$. For each type $i = 1, 2, \dots, k$, denote by $p_i(\mathbf{z})$ the probability that an individual of type i has the offspring vector \mathbf{z} . Note that $\sum_{\mathbf{z}} p_i(\mathbf{z}) = \mathbf{1}$. These probabilities define the *offspring distribution* for individuals of type i .

To analyze the probabilistic structure of the offspring distribution, we can use generating functions. The probability generating functions for a multi-type branching process with the offspring distribution (p_i) is defined as

$$(13) \quad f_i(s_1, \dots, s_k) = \sum_{\mathbf{z} \in \mathbb{N}_k^k} p_i(\mathbf{z}) s_1^{z_1} s_2^{z_2} \cdots s_k^{z_k}, \quad |s_1|, \dots, |s_k| \leq 1.$$

We say f_i is the **probability generating function** for the offspring distribution of individuals of type i .

For a k -type branching process, define the **matrix of means** as the $k \times k$ matrix $\mathbf{M} = [m_{ij}]$ such that

$$(14) \quad m_{ij} = \mathbb{E}[Z_{j1} | \mathbf{Z}_0 = \mathbf{e}_i].$$

In other words, the entry m_{ij} represents the average number of offspring of type j produced by an individual of type i in one generation.

Let us now consider the **multi-type Poisson branching process**, a special case of the multi-type branching process where the number of offspring of each type follows a Poisson distribution. Then the entries of the corresponding mean matrix M are the parameters of the Poisson distributions, i.e. $Z_{ij} \sim \text{Poi}(m_{ij})$. This process has offspring distributions of the form

$$(15) \quad p_i(\mathbf{z}) = \prod_{j=1}^k e^{-m_{ij}} \frac{(m_{ij})^{z_j}}{z_j!}.$$

Applying (15) to (13) and the independence of number of offsprings of each type yields the following form for probability generating function of the multi-type Poisson branching process.

$$(16) \quad f_i(\mathbf{s}) = e^{\langle \mathbf{e}_i | \mathbf{M} | \mathbf{s} - \mathbf{1} \rangle}.$$

2.1. Extinction Probability. As gelation is to vector multiplicative coalescent processes, a fundamental question in the study of branching processes is whether the process becomes *extinct* or not as the number of generations n tends to infinity. Recall that in the classical case, if the expected number of the offspring distribution is less than (or equal to) 1, then eventually the population dies out. But if the expected number is greater than 1, then the probability of extinction is the smallest non-negative solution to the equation $s = \varphi(s)$, where φ is the probability generating function of the corresponding offspring distribution.

For a multi-type branching process with offspring distribution (\mathbf{Z}_n) , let $\boldsymbol{\eta} = (\eta_1, \dots, \eta_k)$ denote the extinction probability vector, where η_i denote the extinction probability of an individual of type i when the process begins with $\mathbf{Z}_0 = \mathbf{e}_i$. In this case, the phase transition for $\boldsymbol{\eta}$ is related to the spectral radius of the mean matrix \mathbf{M} , as stated in the following theorem (see [3]).

Theorem 2.1. *Let $f_i(\mathbf{s})$ be the probability generating function and \mathbf{M} be the matrix of means of a positive regular and non-singular multi-type branching process. Let $\rho(\mathbf{M})$ denote the spectral radius of \mathbf{M} . Furthermore, let $\boldsymbol{\eta}$ be the vector of extinction probabilities.*

(a) *If $\rho(\mathbf{M}) \leq 1$, then $\boldsymbol{\eta} = \mathbf{1}$, the k -dimensional vector $(1, 1, \dots, 1)$.*

(b) *If $\rho(\mathbf{M}) > 1$, then $\boldsymbol{\eta}$ is the smallest (by component) solution less than $\mathbf{1}$ of the fixed point equations*

$$(17) \quad f_i(\mathbf{s}) = s_i, \quad \forall i = 1, 2, \dots, k.$$

In the case of the multi-type Poisson branching process, the fixed point equations (17) have the form

$$(18) \quad e^{\langle \mathbf{e}_i | \mathbf{M} | \mathbf{s} - \mathbf{1} \rangle} = s_i$$

which we will refer to as the *Poisson fixed point equations*.

3. CORRESPONDENCE BETWEEN GELATION AND EXTINCTION PROBABILITY WITH APPLICATIONS

The bridge between the phase transitions of these two stochastic models stems from the equivalence of the multidimensional Lambert-Euler inversion equations (11) and the Poisson fixed point equations (18) with $\mathbf{M} = VD[\alpha_i]t$. Specifically,

$$(19) \quad e^{\langle \mathbf{e}_i | VD[\alpha_i]t | \mathbf{s} - \mathbf{1} \rangle} = s_i \quad \iff \quad y_i e^{-\langle \mathbf{e}_i | V | \mathbf{y} \rangle} = \alpha_i t e^{-\langle \mathbf{e}_i | V | \boldsymbol{\alpha} t \rangle}$$

when $s_i = y_i / (\alpha_i t)$.

Using this equivalence we get our first application that yields the gelation phenomenon for the vector multiplicative coalescent process and the new proof follows immediately from this equivalence, Theorem 2.1 and equation (12).

Theorem 3.1. *Let $\zeta_{\mathbf{x}}(t)$ be the solution to the modified Smoluchowski equations which represents the limiting fraction of clusters of size \mathbf{x} in the vector multiplicative coalescent process.*

- (a) *If $\rho(VD[\alpha_i]t) \leq 1$, then the total mass $\sum_{\mathbf{x}} \zeta_{\mathbf{x}}(t)|\mathbf{x}\rangle = \boldsymbol{\alpha}$.*
 (b) *If $\rho(VD[\alpha_i]t) > 1$, then the total mass $\sum_{\mathbf{x}} \zeta_{\mathbf{x}}(t)|\mathbf{x}\rangle < \boldsymbol{\alpha}$.*

The second application of the equivalence (19) is a new series expression for the extinction probabilities of the Poisson branching process. From the equivalence (19), we have the extinction probability $\eta_{\ell} = y_{\ell}/(\alpha_{\ell}t)$ where y_{ℓ} is the smallest solution to the Lambert-Euler inversion equations. Then by the total mass expression (12) and the explicit solution to the modified Smoluchowski equations (4), we have

$$(20) \quad \eta_{\ell} = \frac{1}{\alpha_{\ell}} \frac{y_{\ell}}{t} = \frac{1}{\alpha_{\ell}} \sum_{\mathbf{x}} \zeta_{\mathbf{x}}(t) x_{\ell} = \frac{1}{\alpha_{\ell}} \sum_{\mathbf{x}} \frac{1}{\mathbf{x}!} \boldsymbol{\alpha}^{\mathbf{x}} \frac{\tau(K_k, x_i x_j v_{i,j})}{\mathbf{x}^{\mathbf{1}}} (V\mathbf{x})^{\mathbf{x}-\mathbf{1}} e^{-\langle \mathbf{x} | V | \boldsymbol{\alpha} \rangle t} t^{\langle \mathbf{x} | \mathbf{1} \rangle - 1} x_{\ell}.$$

From the relation $\mathbf{M} = VD[\alpha_i]t$,

$$\langle \mathbf{x} | V | \boldsymbol{\alpha} \rangle t = \langle \mathbf{x} | \mathbf{M} | \mathbf{1} \rangle$$

and

$$\boldsymbol{\alpha}^{\mathbf{x}} (V\mathbf{x})^{\mathbf{x}-\mathbf{1}} = \frac{\boldsymbol{\alpha}^{\mathbf{1}}}{t^{\langle \mathbf{x} | \mathbf{1} \rangle - k}} (\mathbf{M}\mathbf{x})^{\mathbf{x}-\mathbf{1}}$$

Moreover,

$$\tau(K_k, x_i x_j v_{i,j}) = \frac{1}{t^{k-1}} \tau\left(K_k, x_i x_j \frac{m_{i,j}}{\alpha_j}\right)$$

Substituting the above three equations into (20) yields the following new result for the extinction probabilities of the multi-type Poisson branching process.

Theorem 3.2. *Suppose the matrix of means \mathbf{M} for a multi-type Poisson branching process can be expressed as $\mathbf{M} = VD[\alpha_i]t$ where V is a nonnegative, irreducible and symmetric matrix, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_k) \in (0, \infty)^k$, and t is a nonnegative real number. Then the extinction probabilities of this process $\boldsymbol{\eta} = (\eta_1, \dots, \eta_k)$ have the series form*

$$\eta_{\ell} = \frac{1}{\alpha_{\ell}} \sum_{\mathbf{x}} \frac{\boldsymbol{\alpha}^{\mathbf{1}}}{\mathbf{x}! \mathbf{x}^{\mathbf{1}}} \tau\left(K_k, x_i x_j \frac{m_{i,j}}{\alpha_j}\right) e^{-\langle \mathbf{x} | \mathbf{M} | \mathbf{1} \rangle} x_{\ell}, \quad \forall \ell = 1, 2, \dots, k.$$

Remark 3.3. *Let us note that an additional relationship between vector multiplicative coalescence and multi-type branching processes through Burgers' equation was investigated in a recent paper [10].*

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