Schur's Theorem

Schur's theorem states the following:

Theorem 1. *A finite coloring of* $\mathbb{N}$ *contains* x, y, z*, all the same color, such that* $x+y=z$*.*

Proof. Let $\chi : \mathbb{N} \to \{1, \ldots, r\}$ be an r-coloring of $\mathbb{N}$, and define $N = N(r)$ by the equation

$$N(r) = R(3; r) - 1 := R(3, 3, \ldots, 3) - 1,$$

where there are r 3s in the Ramsey number. We show that we can in fact find the required x, y, z among the integers $1, 2, \ldots, N$.

Let K_{N+1} be the complete graph on vertex set $\{1, 2, \ldots, N+1\}$. Define an edge-coloring χ' of K_{N+1} using the formula

$$\chi'(\{i, j\}) = \chi(|i - j|).$$

Since $N + 1 = R(3; r)$, this K_{N+1} contains a monochromatic triangle, with vertices i, j, k, in which, without loss of generality, $i > j > k$. Setting $x = i-j$, $y = j-k$ and $z = i-k$, we see that x, y and z all get the same color, and that $x+y = (i-j)+(j-k) = i-k = z$. $\square$

Non-examinable historical digression. Schur packed a lot into his 4-page paper [7]. Besides an application to Fermat's Last Theorem (see [2]), he also proved remarkably good bounds on $S(r)$, the smallest N such that an r-coloring of $\{1, 2 \ldots, N\}$ contains (not necessarily distinct) x, y, z with the same color and with $x + y = z$. The above argument (essentially due to Schur) shows that $S(r) \leq N(r) = R(3; r) - 1$. But Schur also [7] proved the bounds

$$\tfrac{1}{2}(3^r + 1) \leq S(r) \leq N(r) \leq F(r) := \lfloor er! \rfloor$$

(in which, yes, $e = 2.71828\ldots$). The lower bound comes from a charming construction (try to find it). For the upper bound, it is not hard to show [4] that $F(r) = rF(r-1) + 1$ for $r \geq 2$. One can then use this (and induction) to prove that $N(r) \leq F(r)$.

r	$\frac{1}{2}(3^r+1)$	$S(r)$	$N(r)$	$F(r)$
1	2	2	2	2
2	5	5	5	5
3	14	14	16	16
4	41	45	$50-61$	65
5	122	161	$161-306$	326

Table 1: Schur and Ramsey numbers

Schur didn't use graphs in his proof. Instead, he expressed the entire argument in its original number-theoretic context, and, of course, in German. For English expositions, see [4] for the graph-theoretic version, and [5] for the number-theoretic version. Remarkably, Schur's asymptotic bounds have hardly been improved in over 100 years [1]. And, for small values of r, his bounds are remarkably good, as the above table shows. The value $S(5)$ was found only in 2017 after a massive computer search [3], while the bounds for $N(4)$ and $N(5)$ are from the recently-updated survey [6].

References

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