# Gelation in Vector Multiplicative Coalescence and Extinction in Multi-Type Poisson Branching Processes

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**Abstract**

In this note, we present a novel connection between a multi-type (vector) multiplicative coalescent process and a multi-type branching process with Poisson offspring distributions. More specifically, we show that the equations that govern the phenomenon of gelation in the vector multiplicative coalescent process are equivalent to the equations that yield the extinction probabilities of the corresponding multi-type Poisson branching process. We then leverage this connection with two applications, one in each direction. The first is a new quick proof of gelation in the vector multiplicative coalescent process, and the second is a new series expression for the extinction probabilities of the multi-type Poisson branching process. Lastly, we describe this connection in the broader context of the relationships between these two processes and random graphs, including a new derivation of the solution to the modified Smoluchowski coagulation equations that govern the dynamics of the vector multiplicative coalescent process.

**Keywords:** Vector-multiplicative coalescent, Gelation, Multitype branching processes, Extinction, Lambert-Euler inversion, Random graphs

## 1. Introduction

The main motivation for the work in this note is the paper [18]. In that paper, the authors introduced the multidimensional Lambert-Euler inversion which is the system of equations given by

$$y_i e^{-\langle \mathbf{e}_i | V | \mathbf{y} \rangle} = \alpha_i t e^{-\langle \mathbf{e}_i | V | \boldsymbol{\alpha} t \rangle} \qquad i = 1, \dots, k.$$

where $V \in \mathbb{R}^{k \times k}$ is a nonnegative irreducible symmetric matrix, $\boldsymbol{\alpha} \in (0, \infty)^k$ and $t \geq 0$. The solutions to the multidimensional Lambert-Euler inversion was then utilized to investigate the gelation phenomenon in the corresponding coagulation model with the vector-multiplicative kernel. Our goal in this note is to present an interesting new connection between the vector multiplicative coalescent process and a multi-type branching process with Poisson offspring distributions through the equivalence of the multidimensional Lambert-Euler inversion equations and the fixed point equations that yield the extinction probabilities in the multi-type branching process. From this equivalence, we see that the phenomenon of *gelation* in the coalescent process is directly related to the phenomenon of *extinction* in the branching process. As applications of this equivalence, we give (a) a new quick proof of the gelation phenomenon in the vector multiplicative coalescent process, and (b) a new series expression for the extinction probabilities of the multi-type Poisson branching process. This illuminates a connection between these two paradigms which we hope will continue to reveal new insights and potential for cross-disciplinary research. To complete the picture, we then describe how this equivalence can be motivated and fits into the broader connections between the three related random processes: coalescence, branching, and random graphs.

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**Outline of the paper.** The rest of the paper is organized as follows: In the next two sections we define the two random processes; the vector multiplicative coalescent process in Section 2 and the multi-type Poisson branching process in Section 3. Included in Section 2 is an introduction of the modified Smoluchowski coagulation equations that govern the dynamics of the vector multiplicative coalescent process and the multidimensional Lambert-Euler inversion result. In Section 4, we present the equivalence of the two sets of equations discussed above and the connections it reveals between the processes. In Section 5, we explain the connections between the coalescent processes and branching processes using the theory of *random graphs*. We then use random graphs to first give an alternative derivation of equation (2) from [18], and next explain the equivalence between the systems of equations expressed in (16).

## 2. Vector multiplicative coalescent processes

We begin by recalling some of the concepts from the theory of vector multiplicative coalescent processes developed in [17, 18]. As it was done in [18], we find it convenient to use bra-ket notation. Specifically, $|\mathbf{x}\rangle$ will denote the column vector representation of vector $\mathbf{x} \in \mathbb{R}^k$, and $\langle \mathbf{x}|$ will denote the row vector representation of vector $\mathbf{x} \in \mathbb{R}^k$. For example, $|\mathbf{1}\rangle$ denotes the $k$-dimensional column vector with all coordinates equal 1. For $c \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^k$, $c|\mathbf{x}\rangle$ will represent the product $c\mathbf{x}$, a column vector. Correspondingly, $\langle \mathbf{x}|\mathbf{y}\rangle = \langle \mathbf{y}|\mathbf{x}\rangle$ will be the dot product of $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^k$. Finally, for a matrix $M \in \mathbb{R}^{k\times k}$, $\langle \mathbf{x}|M|\mathbf{y}\rangle$ will represent the product $\mathbf{x}^{\mathsf{T}} M\mathbf{y}$ resulting in a scalar.

Consider a system with $k$ types of particles: $1, \ldots, k$. The system size is controlled with a large integer parameter $n > 0$. For a given vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in (0,\infty)^k$, the process begins with $\langle \boldsymbol{\alpha} n|\mathbf{1}\rangle + o(\sqrt{n})$ singletons distributed between the $k$ types so that for each $i$, there are $\alpha_i\, n + o(\sqrt{n})$ particles of type $i$. Let $V \in \mathbb{R}^{k\times k}$ be a nonnegative, irreducible and symmetric matrix, which we will refer to as the *partition interaction matrix*. A particle of type $i$ bonds with a particle of type $j$ with the intensity rate $v_{i,j}/n$, where $v_{i,j}$ is the $(i, j)$ element in the matrix $V$. The bonds are formed independently. This process is called the *vector-multiplicative coalescent process*. The vector multiplicative coalescent process describe multiplicative cluster merger dynamics where the weight of each cluster is a vector. As each cluster is composed of particles of different types, each coordinate of its weight vector corresponds to the number of particles of the corresponding type. Note that the irreducibility condition on $V$ ensures that particles of each type could ultimately coalesce with particles of every other type at some time in the vector-multiplicative coalescent process.

For each weight vector $\mathbf{x} \in \mathbb{Z}^k_{\geq 0}$ satisfying $\langle \mathbf{x}|\mathbf{1}\rangle > 0$, let $\zeta_{\mathbf{x}}(t)$ denote the hydrodynamic limit representing the fraction of clusters of size $\mathbf{x}$ at time $t$ as we take $n \to \infty$. The differential equations that govern the dynamics of $\zeta_{\mathbf{x}}(t)$ are the *modified Smoluchowski equations (MSE)* given by

$$\frac{d}{dt}\zeta_{\mathbf{x}}(t) = -\zeta_{\mathbf{x}}(t)\langle \mathbf{x}|V|\boldsymbol{\alpha}\rangle + \frac{1}{2}\sum_{\mathbf{y},\mathbf{z}\,:\mathbf{y}+\mathbf{z}=\mathbf{x}} \langle \mathbf{y}|V|\mathbf{z}\rangle \zeta_{\mathbf{y}}(t)\zeta_{\mathbf{z}}(t) \tag{1}$$

with initial conditions $\zeta_{\mathbf{x}}(0) = \alpha_i$ if $x = \mathbf{e}_i$, and $\zeta_{\mathbf{x}}(0) = 0$ otherwise. In this equation, the summation is over all vectors $\mathbf{y}$ and $\mathbf{z}$ with all $y_i \geq 0, z_i \geq 0$ and with $\sum y_i > 0, \sum z_i > 0$.

**Gelation.** For many kernels, including the multiplicative kernel of this paper, the coalescent process undergoes a phase transition known as *gelation*. Informally, at a certain finite and deterministic time $T_{gel}$, a *gel* forms, containing, as $n \to \infty$, a positive proportion of all the particles. Gelation can only be observed "as $n \to \infty$", since, for fixed $n$, the number of particles is finite, and so every cluster contains a positive fraction of the total number of particles. Mathematically, at $T_{gel}$, two things happen simultaneously [18]:

- The second moments of $\zeta_{\mathbf{x}}(t)$, i.e., the entries in the matrix $\sum_{\mathbf{x}} \zeta_{\mathbf{x}}(t)|\mathbf{x}\rangle\langle \mathbf{x}|$, are no longer finite.
- The limiting total mass $\sum_{\mathbf{x}} \zeta_{\mathbf{x}}(t)|\mathbf{x}\rangle$ is no longer conserved.

Of course, for fixed $n$, mass *is* conserved: gelation can only be observed in the limit $n \to \infty$. But, after $T_{gel}$, the *limiting* mass is not conserved because the gel, invisible in (1), has swallowed some positive fraction of finite-size clusters, and only these clusters are modeled in (1). From now on, we will simply say that mass is not conserved after gelation, and omit reference to the underlying limiting process. For more details, see [15, 21, 3, 23, 12]. Gelation corresponds exactly to the formation of a giant component in an associated random graph, which we describe later.

2.1. **The solution to the modified Smoluchowski equations.** The unique solution $\zeta_{\mathbf{x}}(t)$ of (1) was derived in [18] (see Corollary 3.11). Specifically, for a vector $\mathbf{x} \in \mathbb{Z}^k_{\geq 0}$, let $\mathbf{x}! = x_1!x_2!\dots x_k!$, and for vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^k_+$, let $\mathbf{a}^{\mathbf{b}} = a_1^{b_1}a_2^{b_2}\dots a_k^{b_k}$. Now, consider a complete graph $K_k$ consisting of vertices $\{1,\dots,k\}$ with weights $w_{i,j} = w_{j,i} \geq 0$ assigned to its edges $[i,j]$ $(i \neq j)$. Let the weight $W(\mathcal{T})$ of a spanning tree $\mathcal{T}$ be the product of the weights of all of its edges. Finally, let $\tau(K_k, w_{i,j}) = \sum_{\mathcal{T}} W(\mathcal{T})$ denote the *weighted spanning tree enumerator*, i.e., the sum of weights of all spanning trees in $K_k$. In the case $w_{i,j} \equiv 1$, the weighted spanning tree enumerator simply equals the number of spanning trees of the graph. In terms of these notations, the solution to the modified Smoluchowski equations that yield the limiting fraction of clusters of size $\mathbf{x}$ is given by

$$\zeta_{\mathbf{x}}(t) = \frac{1}{\mathbf{x}!}\boldsymbol{\alpha}^{\mathbf{x}}\frac{\tau(K_k, x_ix_jv_{i,j})}{\mathbf{x}^{\mathbf{1}}}(V\mathbf{x})^{\mathbf{x}-\mathbf{1}}e^{-\langle \mathbf{x}|V|\boldsymbol{\alpha}\rangle t}t^{\langle \mathbf{x}|\mathbf{1}\rangle - 1}. \tag{2}$$

Also see a related derivation using a different approach in [13, equation (3.10)] and in [14, equation (2.9)].

2.2. **Lambert-Euler Inversion.** For the vector multiplicative coalescent process, the initial total mass vector is assumed to be $\sum_{\mathbf{x}} \zeta_{\mathbf{x}}(0)|\mathbf{x}\rangle = |\boldsymbol{\alpha}\rangle$. As discussed above, we define gelation as the loss of total mass $\sum_{\mathbf{x}} \zeta_{\mathbf{x}}(t)|\mathbf{x}\rangle$ of the system after a critical time $T_{gel}$ called the *gelation time*. That is

$$T_{gel} = \inf\Big\{t > 0 \,:\, \sum_{\mathbf{x}} \zeta_{\mathbf{x}}(t)|\mathbf{x}\rangle < |\boldsymbol{\alpha}\rangle\Big\}. \tag{3}$$

In [18, Corollary 4.1], the existence of gelation was proved for the vector multiplicative coalescent process and the gelation time was given by

$$T_{gel} = \frac{1}{\rho(VD[\boldsymbol{\alpha}])}, \tag{4}$$

where $\rho(A)$ denotes the spectral radius (i.e., the largest of the absolute values of the eigenvalues) of the matrix $A$. This result is also a special case of the spectral criticality condition in Corollary 2.3 of [7], and a similar result appears in [3].

The existence of gelation for the vector multiplicative coalescent process was proved by solving a multidimensional Lambert-Euler inversion problem. This is a higher dimensional generalization of the equation originally studied by Lambert and followed up by Euler that gave rise to the well known Lambert W function [19, 10]. For a given vector $\boldsymbol{\alpha} \in (0,\infty)^k$, consider the region

$$R_0 = \left\{\boldsymbol{\alpha} \in (0,\infty)^k \,:\, \rho\left(VD[\boldsymbol{\alpha}]\right) < 1\right\}, \tag{5}$$

its closure within $(0,\infty)^k$,

$$\overline{R}_0 = \left\{\boldsymbol{\alpha} \in (0,\infty)^k \,:\, \rho\left(VD[\boldsymbol{\alpha}]\right) \leq 1\right\}, \tag{6}$$

and the complement of $\overline{R}_0$ within $(0,\infty)^k$,

$$R_1 = \left\{\boldsymbol{\alpha} \in (0,\infty)^k \,:\, \rho\left(VD[\boldsymbol{\alpha}]\right) > 1\right\}. \tag{7}$$

The following theorem implies the existence of a unique solution to the multidimensional Lambert-Euler problem (see [18, Theorem 1.1]).

**Theorem 2.1** (Multidimensional Lambert-Euler inversion). *Consider a nonnegative irreducible symmetric matrix $V \in \mathbb{R}^{k\times k}$. Then, for any given $\boldsymbol{\alpha} \in (0,\infty)^k$ and $t \geq 0$, there exists a unique vector $\mathbf{y} \in \overline{R}_0$ such that*

$$y_i e^{-\langle \mathbf{e}_i|V|\mathbf{y}\rangle} = \alpha_i t e^{-\langle \mathbf{e}_i|V|\boldsymbol{\alpha}t\rangle} \qquad i = 1,\dots,k. \tag{8}$$

*Moreover, if $\boldsymbol{\alpha}t \in \overline{R}_0$, then $\mathbf{y} = \boldsymbol{\alpha}t$. If $\boldsymbol{\alpha}t \in R_1$, then $\mathbf{y} < \boldsymbol{\alpha}t$ ( i.e., $y_i < \alpha_i t$ for all $i$). Informally, $\mathbf{y}$ is the smallest solution of (8), in that each of its components is less than the corresponding component of $\alpha t$.*

Finally, in [18, Lemma 4.2], it was shown that the total mass $\sum_{\mathbf{x}} \zeta_{\mathbf{x}}(t)|\mathbf{x}\rangle$ of the vector multiplicative coalescent process could be expressed in terms of the solution of the multidimensional Lambert-Euler inversion problem as follows:

$$\sum_{\mathbf{x}} \zeta_{\mathbf{x}}(t)|\mathbf{x}\rangle = \frac{1}{t}|\mathbf{y}(t)\rangle. \tag{9}$$

Then, Theorem 2.1 proves the existence of the gelation phenomenon for the vector multiplicative coalescent process and the gelation time given in (4). Specifically,

$$\sum_{\mathbf{x}} \zeta_{\mathbf{x}}(t)|\mathbf{x}\rangle \begin{cases} = |\boldsymbol{\alpha}\rangle & \text{if } t \leq 1/\rho\,(VD[\boldsymbol{\alpha}]) \\ < |\boldsymbol{\alpha}\rangle & \text{if } t > 1/\rho\,(VD[\boldsymbol{\alpha}]). \end{cases}$$

**Remark.** The one dimensional Lambert-Euler inversion reads as

$$x(t) = \min\{x \geq 0 \,:\, xe^{-x} = te^{-t}\}.$$

In their now famous paper [9], Erdős and Rényi used the function $x(t)$ to establish the formation of a giant component in the theory of random graphs. In fact, in the random graph setting, the quantity $x(t)$ can be interpreted as *the average degree outside the giant component* in an Erdős-Rényi graph $G(n, t/n)$. It would be interesting to see if (8) can be used to obtain refined information about the vertices in the giant component of a random inhomogeneous graph, in the same spirit as the classical case.

Note that in this one-dimensional setting, $x(t) = t$ on $(0, 1]$, and $x(t) < t$ on $(1,\infty)$. Moreover, $x(t) \downarrow 0$ monotonically as $t \to \infty$. On the other hand, the post-gelation behavior of the multi-dimensional solution to (8) can differ from the one-dimensional case. For example, consider the bipartite case with $V = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ and $\boldsymbol{\alpha} = (15, 2)$. Then, the critical time is given by $T_{gel} = \frac{1}{\sqrt{30}}$, and for $t > T_{gel}$, the graph of $y(t) = (y_1(t), y_2(t))$ is as follows.

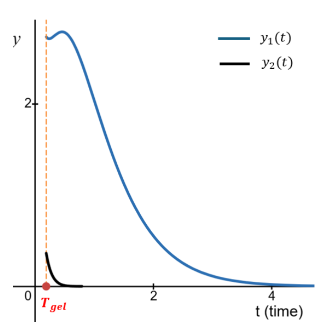


FIGURE 1. Post-gelation behavior of Lambert-Euler inversion

Observe that the function $y_1(t)$ decreases, increases, and finally decreases as $t$ increases. Numerical simulation seems to imply that there is a critical value for the ratio of the parameters in $\boldsymbol{\alpha}$ after which the solutions $(y_1(t), y_2(t))$ are no longer monotonically decreasing. Note, however, that only $(y_1(t)+y_2(t))/t$ has physical significance (as the total mass outside the gel).

## 3. Multi-Type Poisson Branching Processes

In this section, we give an overview of the relevant theory of multi-type branching processes, which is a generalization of the classical Galton-Watson process, extending it to multiple types of individuals. A full theory can be found in [4]. This multi-type model is useful in various fields, including biology, physics, and epidemiology, to study populations with different types of individuals, and how these types evolve over time. Recall that in the classical Galton-Watson process, there is only one type of individual, whereas in the multi-type case, there are $k$ different types: $1, 2, \ldots, k$. We shall assume that the process starts with only one individual of a specific type initially, i.e. $\mathbf{Z}_0 = \mathbf{e}_i$. Let $\mathbf{Z}_n = (Z_{1n}, Z_{2n}, \ldots Z_{kn})$ be the $k$-dimensional vector representing the number of individuals of each type in the $n$th generation of the branching process. Let $\mathbf{z} = (z_1, \ldots, z_k)$. For each type $i = 1, 2, \ldots, k$, denote by $p_i(\mathbf{z})$ the probability that an individual of type $i$ has the offspring vector $\mathbf{z}$. Note that $\sum_{\mathbf{z}} p_i(\mathbf{z}) = \mathbf{1}$. These probabilities define the *offspring distribution* for individuals of type $i$.

To analyze the probabilistic structure of the offspring distribution, we can use generating functions. The probability generating function for a multi-type branching process with the offspring distribution $(p_i)$ is defined as

$$f_i(s_1, \ldots, s_k) = \sum_{\mathbf{z} \in \mathbb{N}_{\mathbf{0}}^{\mathbf{k}}} p_i(\mathbf{z}) s_1^{z_1} s_2^{z_2} \cdots s_k^{z_k}, \qquad |s_1|, \ldots, |s_k| \leq 1. \tag{10}$$

We say $f_i$ is the *probability generating function* for the offspring distribution of individuals of type $i$.

For a $k$-type branching process, define the *matrix of means* as the $k \times k$ matrix $\mathbf{M} = [m_{ij}]$ such that

$$m_{ij} = \mathbb{E}[Z_{j1} \,|\, \mathbf{Z}_0 = \mathbf{e}_i]. \tag{11}$$

In other words, the entry $m_{ij}$ represents the average number of offspring of type $j$ produced by an individual of type $i$ in one generation. In the case where the entries $m_{ij}$ vary, we have a multi-type *inhomogenous* branching process.

Let us now consider the *independent multi-type Poisson branching process*, a special case of the multi-type branching process where the number of offspring of each type of a single individual follow independent Poisson distributions. Then the entries of the corresponding mean matrix $M$ are the parameters of the Poisson distributions. This process has offspring distributions of the form

$$p_i(\mathbf{z}) = \prod_{j=1}^{k} e^{-m_{ij}} \frac{(m_{ij})^{z_j}}{z_j!}. \tag{12}$$

Applying (12) to (10) and the independence of the number of offspring of each type of a single individual yield the following form for probability generating function of the multi-type Poisson branching process:

$$f_i(\mathbf{s}) = e^{\langle \mathbf{e}_i \,|\mathbf{M}\,| \mathbf{s} - \mathbf{1} \rangle}. \tag{13}$$

3.1. **Extinction Probability.** As gelation is to vector multiplicative coalescent processes, a fundamental question in the study of branching processes is whether the process becomes *extinct* as the number of generations $n$ tends to infinity. Recall that in the classical case, if the expected number of the offspring distribution is less than (or equal to) 1, then eventually the population dies out. But if the expected number is greater than 1, then the probability of extinction is the smallest non-negative solution to the equation $s = \varphi(s)$, where $\varphi$ is the probability generating function of the corresponding offspring distribution.

For a multi-type branching process with offspring distribution $(\mathbf{Z}_n)$, let $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_k)$ denote the extinction probability vector, where $\eta_i$ denotes the extinction probability of an individual of type $i$ when the process begins with $\mathbf{Z}_0 = \mathbf{e}_i$. In this case, the phase transition for $\boldsymbol{\eta}$ is related to the spectral radius of the mean matrix $\mathbf{M}$, as stated in the following theorem (see [4, Theorem 4.1]). A matrix $\mathbf{A}$ is *positive regular* if some power $\mathbf{A}^n$ has all entries strictly positive; a branching process is *singular* if each individual has exactly one child, and *nonsingular* otherwise.

**Theorem 3.1.** [4] *Let $f_i(\mathbf{s})$ be the probability generating function and $\mathbf{M}$ be the matrix of means of a positive regular and nonsingular multi-type branching process. Let $\rho(\mathbf{M})$ denote the spectral radius of $\mathbf{M}$. Furthermore, let $\boldsymbol{\eta}$ be the vector of extinction probabilities.*

(a) *If $\rho(\mathbf{M}) \leq 1$, then $\boldsymbol{\eta} = \mathbf{1}$, the $k$-dimensional vector $(1, 1, \ldots, 1)$.*

(b) *If $\rho(\mathbf{M}) > 1$, then $\boldsymbol{\eta}$ is the smallest (by component) solution less than $\mathbf{1}$ of the fixed point equations*

$$f_i(\mathbf{s}) = s_i \,, \ \forall\, i = 1, 2, \ldots, k. \tag{14}$$

In the case of the multi-type Poisson branching process, the fixed point equations (14) have the form

$$e^{\langle \mathbf{e}_i \,| \mathbf{M} \,| \mathbf{s} - \mathbf{1} \rangle} = s_i \tag{15}$$

which we will refer to as the *Poisson fixed point equations*.

## 4. Correspondence between Gelation and Extinction Probability with Applications

The bridge between the phase transitions of these two stochastic models can be seen in the equivalence of the multidimensional Lambert-Euler inversion equations (8) and the Poisson fixed point equations (15) with $\mathbf{M} = VD[\boldsymbol{\alpha}]t$. Specifically,

$$e^{\langle \mathbf{e}_i \,| VD[\boldsymbol{\alpha}]t \,| \mathbf{s} - \mathbf{1} \rangle} = s_i \qquad \Longleftrightarrow \qquad y_i e^{-\langle \mathbf{e}_i | V | \mathbf{y} \rangle} = \alpha_i t e^{-\langle \mathbf{e}_i | V | \boldsymbol{\alpha} t \rangle} \tag{16}$$

when $s_i = y_i/(\alpha_i t)$. The identification $\mathbf{M} = VD[\boldsymbol{\alpha}]t$, or equivalently $m_{ij} = \alpha_j v_{ij} t$, implies that the offspring distribution of the multi-type Poisson branching process (12) has the form

$$p_i(\mathbf{z}) = \prod_{j=1}^{k} e^{-\alpha_j v_{ij} t} \frac{(\alpha_j v_{ij} t)^{z_j}}{z_j!}, \tag{17}$$

where $\mathbf{z} = (z_1, z_2, \ldots, z_k)$.

Define $N(t)$ to be the matrix with row vectors $\mathbf{n}_1(t), \mathbf{n}_2(t), \ldots, \mathbf{n}_k(t)$, where $\mathbf{n}_i(t)$ is the vector-valued "size" of the offspring of an individual of type $i$ at time $t$ such that

$$P(\mathbf{n}_i(t) = \mathbf{z}) = p_i(\mathbf{z}) = \prod_{j=1}^{k} e^{-\alpha_j v_{ij} t} \frac{(\alpha_j v_{ij} t)^{z_j}}{z_j!}.$$

Then the mean vector $E[\mathbf{n}_i(t)] = \mathbf{m}_i = (m_{i1}, m_{i2}, \ldots, m_{ik})$ has components

$$m_{ij} = \alpha_j v_{ij} t.$$

We will call $N(t)$ the (continuous time, multidimensional) *offspring Poisson process* with rate $\mathbf{M} = VD[\boldsymbol{\alpha}]t$.

As a result, the physical connection between the two random processes (vector coalescence and multi-type Poisson branching) yielded from (16) is that the time parameter $t$ in the vector multiplicative coalescent process corresponds to the time parameter of the continuous time, offspring Poisson process $N(t)$.

Consequently, the gelation time $T_{gel} = 1/\rho(VD[\boldsymbol{\alpha}])$ of the vector multiplicative coalescent process yields a phase transition critical value of the extinction probability of the related multi-type Poisson branching process in terms of the offspring Poisson process $N(t)$.

Using this equivalence we get our first application that yields the gelation phenomenon for the vector multiplicative coalescent process discussed in Section 2.2. We state the result again in the theorem below, followed by the short new proof that follows immediately from this equivalence.

**Theorem 4.1.** *Let $\zeta_{\mathbf{x}}(t)$ be the solution to the modified Smoluchowski equations, i.e., $\zeta_{\mathbf{x}}(t)$ represents the limiting fraction of clusters of size $\mathbf{x}$ in the vector multiplicative coalescent process. Then*

(a) *If $\rho(VD[\boldsymbol{\alpha}]t) \leq 1$, then the total mass $\sum_{\mathbf{x}} \zeta_{\mathbf{x}}(t)|\mathbf{x}\rangle = |\boldsymbol{\alpha}\rangle$.*

(b) *If $\rho(VD[\boldsymbol{\alpha}]t) > 1$, then the total mass $\sum_{\mathbf{x}} \zeta_{\mathbf{x}}(t)|\mathbf{x}\rangle < |\boldsymbol{\alpha}\rangle$.*

*Consequently, the gelation time for the vector multiplicative coalescent process is $T_{gel} = \frac{1}{\rho(VD[\boldsymbol{\alpha}])}$.*

*Proof.* Applying the identification $\mathbf{M} = VD[\boldsymbol{\alpha}]t$ to Theorem 3.1 yields

(a) If $\rho(VD[\boldsymbol{\alpha}]t) \leq 1$, then $\boldsymbol{\eta} = \mathbf{1}$, the $k$-dimensional vector $(1, 1, \ldots, 1)$.

*(b)* If $\rho(VD[\boldsymbol{\alpha}]t) > 1$, then $\boldsymbol{\eta}$ is the smallest solution less than $\mathbf{1}$ of the fixed point equations

$$e^{\langle \mathbf{e}_i \,| VD[\boldsymbol{\alpha}]t \,| \mathbf{s}-\mathbf{1}\rangle} = s_i$$

Then the equivalence (16) implies that $\boldsymbol{\alpha}t\boldsymbol{\eta} = \mathbf{y}$, which completes the proof. □

**Remark 4.2.** *In a recent work* [14]*, Hoogendijk et al. described an alternative approach, which relies on establishing the equivalence between the Smoluchowski coagulation equations (see equation (**??**)) and the inviscid Burgers equation using branching process techniques.*

The second application of the equivalence (16) is a new series expression for the extinction probabilities of the Poisson branching process. From the equivalence (16), we have the extinction probability $\eta_\ell = y_\ell/(\alpha_\ell t)$, where $y_\ell$ is the smallest solution to the Lambert-Euler inversion equations. Then by the total mass expression (9) and the explicit solution to the modified Smoluchowski equations (2), we have

$$\eta_\ell = \frac{1}{\alpha_\ell}\frac{y_\ell}{t} = \frac{1}{\alpha_\ell}\sum_{\mathbf{x}} \zeta_{\mathbf{x}}(t) x_\ell = \frac{1}{\alpha_\ell}\sum_{\mathbf{x}} \frac{1}{\mathbf{x}!}\boldsymbol{\alpha}^{\mathbf{x}} \frac{\tau(K_k, x_i x_j v_{i,j})}{\mathbf{x}^{\mathbf{1}}} (V\mathbf{x})^{\mathbf{x}-\mathbf{1}} e^{-\langle \mathbf{x}|V|\boldsymbol{\alpha}\rangle t} t^{\langle \mathbf{x}|\mathbf{1}\rangle - 1} x_\ell. \tag{18}$$

From the relation $\mathbf{M} = VD[\boldsymbol{\alpha}]t$,

$$\langle \mathbf{x}|V|\boldsymbol{\alpha}\rangle t = \langle \mathbf{x}|\mathbf{M}|\mathbf{1}\rangle$$

and

$$(V\mathbf{x})^{\mathbf{x}-\mathbf{1}} = \frac{1}{t^{\langle \mathbf{x}|\mathbf{1}\rangle - k}} (\mathbf{M}D[\alpha_i^{-1}]\mathbf{x})^{\mathbf{x}-\mathbf{1}}$$

Moreover,

$$\tau(K_k, x_i x_j v_{i,j}) = \frac{1}{t^{k-1}}\ \tau\left(K_k, x_i x_j \frac{m_{i,j}}{\alpha_j}\right).$$

Substituting the above three equations into (18) yields the following new result for the extinction probabilities of the multi-type Poisson branching process.

**Theorem 4.3.** *Suppose the matrix of means $\mathbf{M}$ for a multi-type Poisson branching process can be expressed as $\mathbf{M} = VD[\boldsymbol{\alpha}]t$ where $V$ is a nonnegative, irreducible and symmetric matrix, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in$*

*$(0,\infty)^k$, and $t$ is a nonnegative real number. Then the extinction probabilities of this process $\boldsymbol{\eta} = (\eta_1,\dots,\eta_k)$ have the series form*

$$\eta_\ell = \frac{1}{\alpha_\ell}\sum_{\mathbf{x}} \frac{\boldsymbol{\alpha}^{\mathbf{x}}}{\mathbf{x}!\ \mathbf{x}^{\mathbf{1}}}\ \tau\left(K_k, x_i x_j \frac{m_{i,j}}{\alpha_j}\right)(\mathbf{M}D[\alpha_i^{-1}]\mathbf{x})^{\mathbf{x}-\mathbf{1}} e^{-\langle \mathbf{x}|\mathbf{M}|\mathbf{1}\rangle} x_\ell\,, \quad \forall \ell = 1,2\dots,k.$$

In particular, for the homogenous multi-type Poisson branching process where $\boldsymbol{\alpha} = \mathbf{1}$ and $V$ is the matrix with all entries 1, the extinction probabilities have the form

$$\eta_\ell = \sum_{\mathbf{x}} \frac{1}{\mathbf{x}!\ \mathbf{x}^{\mathbf{1}}}\ \tau\left(K_k, x_i x_j t\right) \langle \mathbf{x}|\mathbf{1}\rangle^{\langle \mathbf{x}|\mathbf{1}\rangle - k} t^{\langle \mathbf{x}|\mathbf{1}\rangle - k} e^{-\langle \mathbf{x}|\mathbf{1}\rangle k t} x_\ell\,, \quad \forall \ell = 1,2\dots,k. \tag{19}$$

Note that equation (19) is consistent with the single-type case. For example, consider the process with two types $\mathbf{x} = (x_1, x_2)$. Suppose $n = x_1 + x_2$. Then, (19) simplifies to

$$\begin{aligned}
\eta_1 &= \sum_{x_1\geq 1, n\geq 1} \frac{t}{x_1!\,(n-x_1)!}(tn)^{n-2} e^{-2tn} x_1 \\
&= \sum_{n\geq 1}\sum_{x_1=1}^{n-1} \frac{t}{x_1!\,(n-x_1)!}(tn)^{n-2} e^{-2tn} x_1 \\
&= \sum_{n\geq 1} t(tn)^{n-2} e^{-2tn}\left(\sum_{x_1=1}^{n-1} \frac{1}{x_1!\,(n-x_1)!} x_1\right).
\end{aligned}$$

The inner sum can be rewritten as

$$\frac{1}{(n-1)!}\sum_{x_1=1}^{n-1} \frac{(n-1)!}{(x_1-1)!\,(n-x_1)!}$$

which by the binomial theorem, is equivalent to $\dfrac{2^{n-1}}{(n-1)!}$. Therefore,

$$\begin{aligned}
\eta_1 &= \sum_{n\geq 1} t(tn)^{n-2} e^{-2tn} \frac{2^{n-1}}{(n-1)!} \\
&= \sum_{n\geq 1} \frac{n^{n-1}}{n!}(2t)^{n-1} e^{-2tn}.
\end{aligned}$$

Letting $z = 2te^{-2t}$, the last summation becomes

$$\eta_1 = \frac{1}{2t}\sum_{n\geq 1} \frac{n^{n-1}}{n!} z^n.$$

Let $T(z) = \sum_{n\geq 1} \dfrac{n^{n-1}}{n!} z^n$, which is known as the *tree function* and it satisfies the functional equation $T(z)e^{-T(z)} = z$ and so, $T(z) = -W(-z)$, where $W$ is the *Lambert W function* (see [8, section 2]). Therefore, $\eta_1 = -\frac{1}{2t}W(-2te^{-2t})$.

On the other hand, for a single-type Poisson branching process with mean $\mu$, the extinction probability $q$ is given by the fixed point equation

$$q = e^{\mu(q-1)}.$$

Equivalently, $-\mu q = W(-\mu e^{-\mu})$, which implies $q = -\frac{1}{\mu}W(-\mu e^{-\mu})$. Hence, $\eta_1$ can be viewed as the extinction probability of a Poisson branching process with mean $\mu = 2t$, so if $\mu \le 1$ (i.e. $t \le 1/2$), then extinction is definite; $q = 1$ (i.e. $\eta_1 = 1$).

## 5. Random graphs and an alternative proof of equation (2)

In this section, we explain the connection between the vector multiplicative coalescent process and the theory of *random graphs*. Random graphs model networks with random connections. They were introduced by Erdős and Rényi in 1960 [9]. The basic model $G(n,p)$ has $n$ vertices, with edges inserted independently at random with probability $p$. One way to think of this model is to imagine $n$ fixed and very large, with $p$ slowly increasing. Specifically, we will set $p = t/n$, where $t$ is a time parameter, so that, as time progresses, the graph $G = G(n,t/n)$ gathers more and more edges, progressing from a collection of isolated vertices to the complete graph with all edges present. Along the way, $G$ undergoes several *phase transitions*. One of the most interesting is the emergence of the so-called *giant component* at $p = 1/n$, i.e., at time $t = 1$. Roughly speaking, at fixed time $t = 1-\epsilon$, as $n \to \infty$, all the connected components of $G$ have size $O(\log n)$, but at time $t = 1+\epsilon$, the largest connected component of $G$ (the "giant") has size $C(\epsilon)n$. As with gelation, the emergence of the giant component is an asymptotic phenomenon, which can only be observed in the limit as $n \to \infty$. It was studied in detail by Erdős and Rényi [9]; subsequent work by Bollobás and others led to the field of random graphs [6].

The connection between the vector multiplicative coalescent process $C$ and (an inhomogeneous version of) the above random graph $G$ works as follows. Small clusters of particles in $C$ correspond to small connected components of $G$. The giant component of $G$ corresponds to the gel in $C$. The correspondence between $C$ and $G$ was first described in [1] for the Erdős-Rényi model (which corresponds to a coalescent process with only one type of particle), and in [18, Section 1.2] for the vector multiplicative process with $k$ particle types. We refer to [18] for a full account. Here, we include only enough detail to describe the proof of (2).

Our random graph $G$, whose $n$ vertices are partitioned into $k$ sets $A_1, \dots, A_k$, models the coalescent process at a fixed time $t$. The $n$ particles in $C$ correspond to the $n$ vertices of $G$, and, as before, we will let $n \to \infty$. The numbers $\alpha_i n$ of particles of each type in $C$ correspond to the sizes of the parts $A_i$ of $G$, so that $G$ consists of $n$ vertices, divided into $k$ parts $A_1, \dots, A_k$, in such a way that part $A_i$ contains $\alpha_i n$ vertices. Each potential edge $x_i x_j$ of $G$ (with $x_i \in A_i$ and $x_j \in A_j$) is included in $E(G)$, the edge set of $G$, with probability

$$1 - e^{-v_{ij}t/n} = \frac{v_{ij}t}{n} + O\left(\left(\frac{v_{ij}t}{n}\right)^2\right).$$

Here, $V = (v_{ij})$ is the partition interaction strength matrix defined above. The exponential formulation is convenient (but not strictly necessary) so that $G$ can be seen as a snapshot of a certain memoryless process. Finally, for our purposes, $t$ will be a constant, not depending on $n$. Note that if $k = \alpha_1 = v_{11} = 1$ then $G$ is (modulo the approximation above) just the Erdős-Rényi graph $G(n,t/n)$, with average degree (approximately) $t$. (Also note that we are allowing edges inside the parts, so that our graphs are not necessarily $k$-partite.)

With $\boldsymbol{\alpha}$ and $V$ fixed, there will be a critical value $t'$ of $t$ below which there is almost surely no giant component, and above which there is almost surely a giant component. Denote by $D[\boldsymbol{\alpha}]$ the diagonal matrix with the $\alpha_i$ on the diagonal. Then, this critical value $t'$ is such that the largest eigenvalue of the matrix $t'VD[\boldsymbol{\alpha}]$ equals 1 (i.e., $t'$ is the reciprocal of the spectral radius of $VD[\boldsymbol{\alpha}]$) [18].

Next, again with $V$ and $\boldsymbol{\alpha}$ fixed, and for given values of $\mathbf{x} = (x_1, \dots, x_k)$ and $t$, we seek the scaled number of *finite* components in $G$ containing $x_i$ vertices of type $i$. By "scaled", we mean the number of such components divided by $n$. Call (the limit as $n \to \infty$ of) this scaled quantity $\phi_{\mathbf{x}}(t)$. Note that the process undergoes a qualitative change when $t = t'$, but, for a given $\mathbf{x}$, the functions $\phi_{\mathbf{x}}(t)$ are in fact continuous for all $t$ [6, 18]. Informally, when it comes to counting *components* instead of vertices, the giant component is asymptotically irrelevant, since there is only one of it, and we are dividing by $n$.

The following theorem, connecting the process $C$ and the graph $G$, appears in [18] (see also [1]).

**Theorem 5.1.** [18] *For given $\boldsymbol{\alpha}, V$ and for any $\mathbf{x}$ and $t < t' = T_{\text{gel}}$, we have*

$$\phi_{\mathbf{x}}(t) = \zeta_{\mathbf{x}}(t),$$

*where the functions $\zeta_{\mathbf{x}}(t)$ are the solutions of the modified Smoluchowski equations* (1) *above.*

We refer the reader to [18] for the proof of Theorem 5.1 and the references therein. In the following, we use Theorem 5.1 to give an alternative proof of (2).

*Alternative proof of* (2). Theorem 5.1 enables us to use the random graph formulation, described above, to compute $\zeta_{\mathbf{x}}(t)$ in the pre-gelation period. We choose a method described to prove [2, Theorem 11.4.2 ].

For simplicity, we first address the case where $k = \alpha_1 = v_{11} = 1$. In this case, setting $\mathbf{x} = l$, we wish to show that

$$\zeta_l(t) = \phi_l(t) = \frac{l^{l-2}t^{l-1}e^{-lt}}{l!}. \tag{20}$$

We calculate $p_l(t) = l\phi_l(t)$, which is the probability that a given vertex lies in a connected component $G_l$ of size $l$. Crucial to this calculation is the fact that, when $p = t/n$ with $n \to \infty$, almost all such connected components are *trees*, i.e., they contain no cycles. This is because, for a fixed set $L$ of $l$ vertices, we require $l-1$ edges to connect the vertices in $L$, and so some such set of $l-1$ edges is present in $G(n,p)$ with probability $q_l = C_l p^{l-1}$. But to connect the vertices in $L$ while also creating a cycle in $L$, we require an additional edge, making $l$ edges in total, and the probability of this is $C'_l p^l = o(q_l)$ as $n \to \infty$, since $p = t/n$. A full proof is given in [6].

Now, setting $p = t/n$ for the edge probability, we calculate the probability that a vertex $v$ lies in some component $G_l$ with $l$ vertices. For this to happen, the vertices in $G_l$ have to be connected to each other, and also none of them can be connected to any vertex outside $G_l$. There are $\binom{n-1}{l-1}$ choices for the other vertices in $G_l$, and, if they are all connected to each other, the edges connecting them form a tree, with probability tending to 1 as $n \to \infty$ (see above). There are $l^{l-2}$ labeled trees on these chosen vertices, and each of these trees is present in $G(n,p)$ with probability $p^{l-1}(1-p)^{\binom{l}{2}-l+1} \sim p^{l-1}$. The probability that two such trees are present is again negligible, since this only happens when $G_l$ contains a cycle. Consequently, we can treat the events indicating the presence of different spanning trees of $L$ in $G(n,p)$ as *disjoint*, so that the probability of their union will asymptotically equal the sum of their individual probabilities, which is asymptotically $l^{l-2}p^{l-1}$. Finally, the edges between $G_l$ and the other vertices are missing with probability $(1-p)^{l(n-l)}$.

Putting this all together, as $n \to \infty$ with $p = t/n$, we see that

$$p_l(t) \sim \binom{n-1}{l-1} \cdot l^{l-2} \cdot p^{l-1} \cdot (1-p)^{l(n-l)} \sim \frac{n^{l-1}}{(l-1)!} \cdot l^{l-2} \cdot \left(\frac{t}{n}\right)^{l-1} \cdot e^{-pln}$$
$$= \frac{1}{(l-1)!} \cdot l^{l-2} \cdot t^{l-1} \cdot e^{-tl} = \frac{l^{l-1} t^{l-1} e^{-lt}}{l!},$$

which, recalling that $p_l(t) = l\phi_l(t)$, establishes (20).

Next we consider the general case. For this we once again require the *weighted spanning tree enumerator* defined above, although here it arises in a slightly different form. Write $K_{\mathbf{x}}(V)$ for the complete graph on $\langle \mathbf{x}|\mathbf{1}\rangle$ vertices with $x_i$ vertices of type $i$, and where the weight of an edge between vertices of types $i$ and $j$ is $v_{ij}$. Suppressing the dependence on $V$, write $T_{\mathbf{x}}$ for the sum of the weights of all the spanning trees of $K_{\mathbf{x}}(V)$. Then, from Theorem 3.8 in [18],

$$T_{\mathbf{x}} = \frac{\tau(K_k, x_i x_j v_{i,j})}{\mathbf{x}^{\mathbf{1}}} (V\mathbf{x})^{\mathbf{x}-\mathbf{1}}.$$

Now, generalizing the argument above, pick a random vertex $v$. The probability that $v$ is of type $i$ is $\alpha_i$. For simplicity of notation, suppose that $i = 1$. We aim to compute $p_{\mathbf{x}}(t)$, which is the probability that $v$ lies in a connected component $G_{\mathbf{x}}$ of type $\mathbf{x}$. Again, with probability tending to 1, $G_{\mathbf{x}}$ is a tree [6]. Up to negligible discrepancies arising from floor and ceiling functions, this probability is obtained by combining three contributions: the number of ways to choose the remaining vertices in $G_{\mathbf{x}}$, given by $\binom{\alpha_1 n}{x_1 - 1}\binom{\alpha_2 n}{x_2} \cdots \binom{\alpha_k n}{x_k}$; the probability that these vertices form a tree, which is asymptotically $T_{\mathbf{x}} \cdot (t/n)^{\langle \mathbf{x}|\mathbf{1}\rangle - 1}$; and the probability that no edges connect $G_{\mathbf{x}}$ to the rest of the graph, which is asymptotically $\exp(-\sum_i \sum_j x_i v_{ij} \alpha_j t)$.

Therefore, as $n \to \infty$,

$$p_{\mathbf{x}}(t) \sim \langle \mathbf{x}|\mathbf{1}\rangle \frac{n^{\langle \mathbf{x}|\mathbf{1}\rangle - 1} \alpha_1^{x_1} \cdots \alpha_k^{x_k}}{x_1! \cdots x_k!} \cdot T_{\mathbf{x}} \cdot \left(\frac{t}{n}\right)^{\langle \mathbf{x}|\mathbf{1}\rangle - 1} \cdot \exp\left(-\sum_i \sum_j x_i v_{ij} \alpha_j t\right)$$
$$= \langle \mathbf{x}|\mathbf{1}\rangle \frac{\alpha_1^{x_1} \cdots \alpha_k^{x_k}}{x_1! \cdots x_k!} \cdot T_{\mathbf{x}} \cdot t^{\langle \mathbf{x}|\mathbf{1}\rangle - 1} \cdot \exp\left(-\sum_i \sum_j x_i v_{ij} \alpha_j t\right),$$

so that, since

$$p_{\mathbf{x}}(t) = \phi_{\mathbf{x}}(t) \langle \mathbf{x}|\mathbf{1}\rangle,$$

we have

$$\zeta_{\mathbf{x}}(t) = \phi_{\mathbf{x}}(t) = \frac{\alpha_1^{x_1} \cdots \alpha_k^{x_k}}{x_1! \cdots x_k!} \cdot T_{\mathbf{x}} \cdot t^{\langle \mathbf{x}|\mathbf{1}\rangle - 1} \cdot \exp\left(-\sum_i \sum_j x_i v_{ij} \alpha_j t\right),$$

proving (2). □

**Remark 5.2 (The random graphs perspective on equation (16)).** *Random graphs may also be used to show that the correspondence between the coalescent process $C$ and the branching process $B$, encapsulated in (16), is not just through equations. There is in fact an asymptotic coupling of $C$, $B$ and an inhomogeneous random graph $G$. The connection between $C$ and $G$ is described in [1] and [18], and the connection between $G$ and $B$ was rigorously established in [7]. Here, we just outline the connections and make some remarks.*

*A given cluster $X$, in the coalescent process $C$, at time $t$, containing $m$ particles, was formed from $m-1$ distinct mergers of sub-clusters before time $t$. For the multiplicative kernel only, such mergers can be thought of as mergers between individual particles in the sub-clusters, and so together they correspond to a spanning tree $T_X$ among the particles in $X$. Such a spanning tree is illustrated in the first panel in Figure 2. $T_X$ corresponds to a connected component in an inhomogeneous random graph $G$. The part sizes and (matrix of) edge probabilities in $G$ correspond to the mass of particle types and particle interaction matrix of $C$: see Table 1 for the details. When $t$ is a constant, almost all the connected components of $G$ will be trees.*

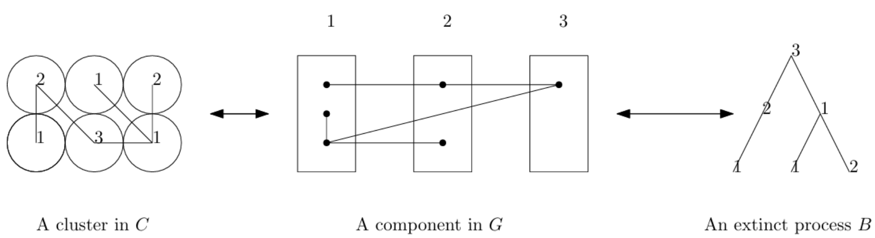


FIGURE 2. Three coupled models: $C$, $G$ and $B$

*Next, we single out a vertex $v \in T_X$, of type $i$. (In Figure 2, $i = 3$.) We can "explore" the component of $G$ containing $v$ using a multi-type Poisson branching process $B$. In $B$, the parameters $t, \alpha_i$ and $v_{ij}$ have no direct interpretation, but they all factor into the means of the Poisson offspring distributions. Increasing $t$ increases the expected number $m_{ij} = v_{ij}\alpha_j t$ of offspring of each type $j$, from each type $i$, in a linear fashion: so, as $t$ increases, the branching process gets "wider", not "deeper". In particular, the branching process $B$ is not the time-reversed coalescent process $C$, the parameter $t$ is not the usual time parameter in the branching process, and the $n$th generation of $B$ does not correspond to the number of particles in $C$. However, the extinction probability $s_i(t)$ does correspond to the probability that a particle of type $i$ in $C$ has not joined the gel by time $t$.*

*For the branching process $B$, the time parameter has an interpretation in terms of the offspring distribution of a single parent. For a parent of type $i$, and offspring of type $j$, we construct a one-dimensional Poisson process $P$ on $\mathbb{R}^+$ of rate $\alpha_{ij}v_j$. For a given time $t$, the random number $N_{ij}(t)$ of offspring (of type $j$ produced by the parent of type $i$) is just the number of events in $P$ up to time $t$, i.e., in the interval $[0, t]$.*

| Quantity | Interpretation in $C$ | Interpretation in $G$ | Interpretation in $B$ |
|---|---|---|---|
| $t$ | time | time | expected offspring number in $B$ |
| $\alpha_i$ | proportion of particles of type $i$ | proportion of vertices of type $i$ | see text |
| $v_{ij}$ | $n \cdot$ bonding rate for types $i$ and $j$ | $(n/t)\cdot$ edge probability for types $i$ and $j$ | see text |
| $s_i(t)$ | $\mathbb{P}$(particle of type $i$ not in gel) | $\mathbb{P}$(vertex of type $i$ not in giant) | extinction probability for type $i$ |
| $v_{ij}\alpha_j t$ | – | – | $\mathbb{E}$(type $j$ offspring from type $i$) |
| $\alpha_i s_i(t)$ | proportion of type $i$ not in gel | proportion of type $i$ not in giant | – |

Table 1. The interpretation of various quantities in the different models

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