Ramsey's Theorem

There are many versions of Ramsey's theorem. Perhaps the simplest is the following. Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of natural numbers, and, for a set $A \subset \mathbb{N}$, let A^2 denote the set of unordered pairs of members of A. We will abuse notation slightly and write, for instance, $A^2 = \{12, 13, 23\}$ if $A = \{1, 2, 3\}$. A *finite coloring* of a set A is technically a map from A to some finite set of colors, but we will think of it as simply a coloring of the elements of A; an r-coloring is a coloring in which (potentially) r colors are used. Finally, a coloring of a set A is *monochromatic* on $B \subset A$ if every $b \in B$ receives the same color. Ramsey's theorem states the following.

Theorem 1. *A 2-coloring of* $\mathbb{N}^2$ *contains an infinite set* A *such that* A^2 *is monochromatic.*

Proof. Consider a given 2-coloring of $\mathbb{N}^2$, with colors red and blue, say, and choose $a_1 \in \mathbb{N}$. There are infinitely many pairs a_1b, each colored either red or blue, so there must be an infinite set B_1 such that the color of each a_1b_1 (for $b_1 \in B_1$) is the same – red, say. Choose $a_2 \in B_1$. There are infinitely many pairs a_2b, for $b \in B_1$, and each is colored either red or blue, so there must be an infinite set $B_2 \subset B_1$ such that the color of each a_2b_2 (for $b_2 \in B_2$) is the same – blue, say. Continue. We obtain an infinite set $A' = \{a_1, a_2, \ldots\}$ in which the color of a_ia_j, where $i < j$, depends only on i: call that color c_i. Each c_i is either red or blue, so there must be a subsequence c_{i_j} such that all the c_{i_j} are the same color, say blue. But then $A = \{a_{i_1}, a_{i_2}, \ldots\}$ is our desired monochromatic set. $\square$

There are many ways to generalize this theorem. One is to increase the number of colors from 2 to r. An identical conclusion holds, and the same proof works. One can also deduce the r-color version directly from Theorem 1, by grouping the colors. Specifically, given a 3-coloring of $\mathbb{N}^2$ with colors red, blue and green, recolor everything colored either red or blue with color purple. Now there are only two colors, purple and green, so we can apply Theorem 1 to get an infinite $B \subset \mathbb{N}$ with B^2 monochromatic. If B^2 is green, we are done. If B^2 is purple (i.e., a red/blue mix), we just apply Theorem 1 again, with the original colors red and blue, to get $A \subset B$ with A^2 monochromatic. The extension to r colors uses the same idea.

A more sophisticated generalization involves coloring the k-sets of $\mathbb{N}$ (i.e., the subsets of $\mathbb{N}$ of size k). Write A^k for the set of subsets of A of size k, and write, for instance, $A^3 = \{123, 124, 134, 234\}$ if $A = \{1, 2, 3, 4\}$. Here is the generalization.

Theorem 2. *A 2-coloring of $\mathbb{N}^k$ contains an infinite set A such that A^k is monochromatic.*

Proof. We modify the proof of Theorem 1. The main new idea is to use induction on k, and to apply the induction hypothesis at each step. The case $k = 1$ is just the "infinite pigeonhole principle", and the case $k = 2$ is Theorem 1.

Consider then a given 2-coloring of $\mathbb{N}^k$, with colors red and blue, say, and choose $a_1 \in \mathbb{N}$. There are infinitely many k-sets $\{a_1\} \cup X$, with $|X| = k - 1$, each colored either red or blue, so, by induction, there must be an infinite set B_1 such that the color of each $\{a_1\} \cup X_1$ (for $X_1 \in B_1^{k-1}$) is the same – red, say. Choose $a_2 \in B_1$. There are infinitely many k-sets $\{a_2\} \cup X$, for $X \in B_1^{k-1}$, and each is colored either red or blue, so, by induction, there must be an infinite set $B_2 \subset B_1$ such that the color of each $\{a_2\} \cup X_2$ (for $X_2 \in B_2^{k-1}$) is the same – blue, say. Continue. We obtain an infinite set $A' = \{a_1, a_2, \ldots\}$ in which the color of $a_{i_1} a_{i_2} \cdots a_{i_k}$, where $i_1 < i_2 < \cdots < i_k$, depends only on $i = i_1$: call that color c_i. The rest of the proof is the same as before: each c_i is either red or blue, so there must be a subsequence c_{i_j} such that all the c_{i_j} are the same color, say blue. But then $A = \{a_{i_1}, a_{i_2}, \ldots\}$ is our desired monochromatic set. □

There are finite versions of these theorems as well. Write $[n] = \{1, 2, \ldots, n\}$. Then, for all t, there exists an $R(t)$ such that every 2-coloring of $[R(t)]^2$ contains A with $|A| = t$ and A^2 monochromatic. This can be proved using the ideas above, and it can also be deduced directly from Theorem 1 by "compactness". Compactness works like this: suppose there is *no* $R(t)$ such that every 2-coloring of $[R(t)]^2$ contains A of size t with A^2 monochromatic. Then, for each $n \geq t$, there is a 2-coloring χ_n of $[n]^2$ without a monochromatic A^2 (where $|A| = t$). Now, list the elements of $\mathbb{N}^2$ in some order, say the *colexicographic order* $12, 13, 23, 14, 24, 34, 15, 25, 35, 45, 16, \ldots$. Among the 2-colorings χ_n, there is an infinite subsequence of colorings in which 12 always gets the same color – red, say. Then, among the colorings in that subsequence, there is a sub-subsequence in which 13 always gets the same color – blue, say. We keep taking subsequences, obtaining in the process a 2-coloring χ of $\mathbb{N}^2$ (in our example, $\chi(12)$ is red and $\chi(13)$ is blue). But this coloring χ contradicts Theorem 1: it does not even contain a monochromatic A^2 with $|A| = t$.

Compactness is a useful technique; it allows us to deduce finite versions of Ramsey-theoretic results from their infinite counterparts. It does not work in reverse.