# Sums, Differences and Dilates

Amites Sarkar

Western Washington University

6 April 2026

Joint work with Jon Cutler (Montclair State University)
and Luke Pebody

# Jon and Luke

# Sums and differences

For a set $A \subset \mathbb{Z}$, the sumset $A + A$ of $A$ is defined as

$$A + A = \{a + b : a, b \in A\}$$

and the difference set $A - A$ is defined as

$$A - A = \{a - b : a, b \in A\}.$$

Example: If $A = \{1, 2, 3\}$, then $A + A = \{2, 3, 4, 5, 6\}$, and $A - A = \{-2, -1, 0, 1, 2\}$.

Example: If $A = \{1, 2, 4\}$, then $A + A = \{2, 3, 4, 5, 6, 8\}$, and $A - A = \{-3, -2, -1, 0, 1, 2, 3\}$.

$A = \{1, 2, 4\}$ is an example of a Sidon set, i.e., a set with no nontrivial equality of the form $a + b = c + d$.

# Number-theoretic connections

**Goldbach's Conjecture**

If $P$ is the set of primes, and $E = \{4, 6, 8, 10, \ldots\}$, then $E \subset P + P$.

**Fundamental Theorem of Arithmetic**

For every $n \geq 2$, there is a unique representation

$$\log n = n_1 \log p_1 + \cdots + n_r \log p_r.$$

- What does this, by itself, imply about the $p_i$?
- Erdős (1938): It implies that $\sum_{p \in P} 1/p$ diverges.

# Freiman's theorem

## Observation

If $A$ is an arithmetic progression, then $|A+A| = 2|A| - 1$.

## Definition

A generalized arithmetic progression (GAP) of dimension $d$ is a set of the form

$$A = \{a_0 + n_1 a_1 + \cdots + n_d a_d\}$$

where $a_i, n_i \in \mathbb{Z}$ and $0 \leq n_i < N_i$. The size of $A$ is $N_1 N_2 \cdots N_d$.

## Freiman's Theorem (1966)

If $A \subset \mathbb{Z}$ and $|A+A| \leq K|A|$, then $A$ is a subset of a GAP of dimension at most $d(K)$ and size at most $s(K)|A|$.

Here, $d(K)$ and $s(K)$ depend only on $K$.

• In words, if $A$ has small doubling, then $A$ must be a dense subset of the projection of a low-dimensional discrete box.

# From a book review

Freiman's original proof was very complicated and published as a book.

From Gordon's review of the English translation of Freiman's book:

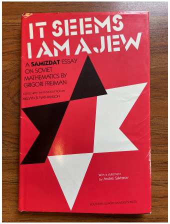
A rather grandiose statement is made to the effect that additive number theory is the study of properties invariant under isomorphism, and comparisons with Klein's *Erlanger Programm* are drawn. The reviewer finds this quite ludicrous, but it does enliven some otherwise dull moments in reading the book.

(On Freiman's proof) The proof of this theorem is a *tour de force*, combining the method of trigonometric sums with ideas from the geometry of numbers and probability theory.

At the risk of reopening the cold war, the reviewer feels compelled to lodge a feeble protest against the sloppiness which mars so many otherwise superb Russian texts. G. H. Hardy once remarked that no one ever wrote five pages of mathematics without a mistake; it seems that in the area of trigonometric sums this bound can be reduced to five lines.

# Gregory Abelevich Freiman (1926-2024)

Freiman (bottom left)




Freiman's other book

• Freiman's book *It Seems I Am a Jew* describes the discrimination against Jewish mathematicians at the Steklov Institute in Moscow, and other institutions, during the postwar period.

• In the 1970s, only 2 to 4 Jewish students were admitted out of a class of 400 to 500 mathematics students at Moscow University.

• Jewish candidates had to sit a special entrance examination, with problems selected from Olympiad competitions.

• Freiman eventually left the USSR and settled in Israel.

# Part of Freiman's legacy

• Freiman's theorem was reproved by Ruzsa in 1994, with a much more streamlined proof.

• The theorem, and Ruzsa's proof of it, were key ingredients in Gowers's 2001 famous proof of Szemerédi's theorem.

• Very recently, a major generalization of Freiman's theorem, the *Polynomial Freiman-Ruzsa conjecture*, also known as *Marton's conjecture*, was settled affirmatively by Gowers, Green, Manners and Tao. The proof was formalized in Lean 4.

• Two books: *Additive Combinatorics* by Terence Tao and Van Vu, and *Graph Theory and Additive Combinatorics* by Yufei Zhao.

# Back in the USSR

**Theorem (Freiman and Pigarev 1973)**

$$|A+A|^{3/4} \leq |A-A| \leq |A+A|^{4/3}$$

**Theorem (Ruzsa 1976, 1989)**

$$\left(\frac{|A+A|}{|A|}\right)^{\frac{1}{2}} \leq \frac{|A-A|}{|A|} \leq \left(\frac{|A+A|}{|A|}\right)^{2}$$

The upper bound comes from taking $B = C = -A$ in Ruzsa's triangle inequality:

$$|A||B-C| \leq |A-B||A-C|,$$

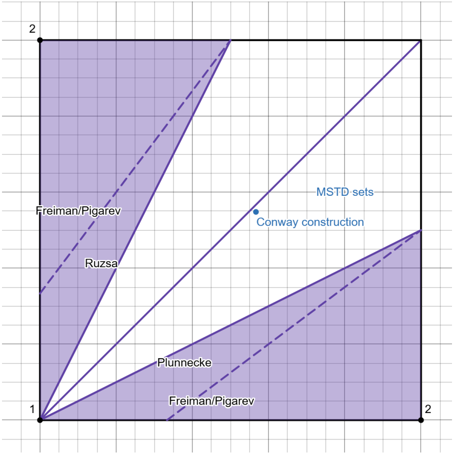
and the lower bound comes from taking $B = C = -A$ in this inequality:

$$|A||B+C| \leq |A+B||A+C|.$$

**Theorem (Conway 1967)**

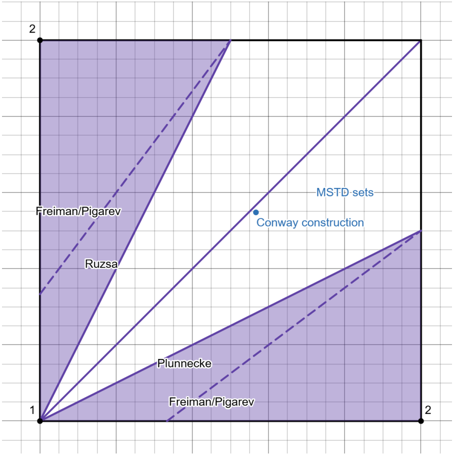
If $A = \{0, 2, 3, 4, 7, 11, 12, 14\}$, then $|A+A| = 26 > 25 = |A-A|$.

# The feasible region $F_{1,-1}$



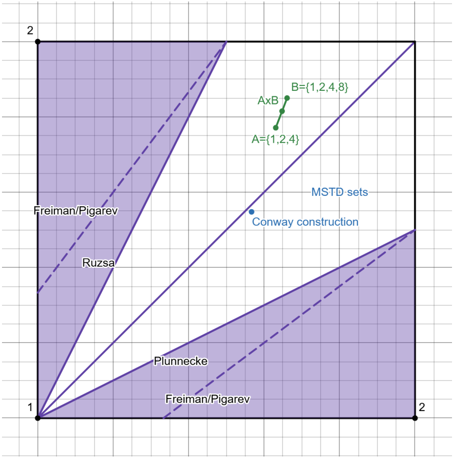

$$E_{1,-1} = \{(x,y) : |A+A| = |A|^x, |A-A| = |A|^y \text{ for some } A \subset \mathbb{Z}\}$$

# The feasible region $F_{1,-1}$




$$E_{1,-1} = \left\{ \left( \frac{\log |A+A|}{\log |A|}, \frac{\log |A-A|}{\log |A|} \right) \right\}, F_{1,-1} = \overline{E_{1,-1}}$$

# The feasible region $F_{1,-1}$ is convex




$$E_{1,-1} = \left\{ \left( \frac{\log|A+A|}{\log|A|}, \frac{\log|A-A|}{\log|A|} \right) \right\}, F_{1,-1} = \overline{E_{1,-1}}$$

# The top right corner of $F_{1,-1}$

## Observation

If $A \subset \mathbb{Z}$ with $|A| = n$, then

$$|A + A| \leq \frac{n(n+1)}{2} \qquad \text{and} \qquad |A - A| \leq n^2 - n + 1,$$

with equality in both cases precisely when $A$ is a Sidon set.

## Theorem (Ruzsa 1992)

For every large enough $n$, there is a set $A$ with $|A| = n$,

$$|A + A| \leq n^{2-c} \qquad \text{and} \qquad |A - A| \geq n^2 - n^{2-c},$$

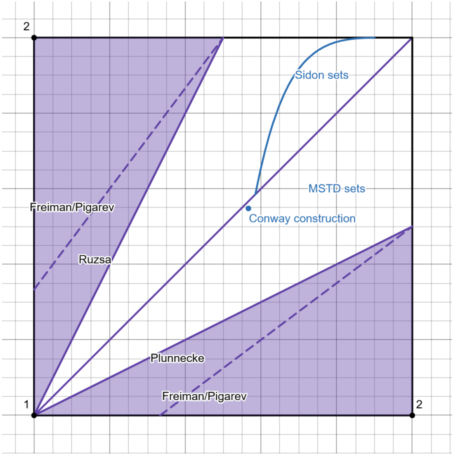
where $c$ is a positive absolute constant, and a set $B$ with $|B| = n$,

$$|B - B| \leq n^{2-c'} \qquad \text{and} \qquad |B + B| \geq \tfrac{1}{2}n^2 - n^{2-c'},$$

where $c'$ is a positive absolute constant.

So $(2 - c, 2) \in F_{1,-1}$ and $(2, 2 - c') \in F_{1,-1}$, for some $c, c' > 0$.

# The feasible region $F_{1,-1}$ and Sidon sets




$$E_{1,-1} = \left\{ \left( \frac{\log|A+A|}{\log|A|}, \frac{\log|A-A|}{\log|A|} \right) \right\}, F_{1,-1} = \overline{E_{1,-1}}$$

# Ruzsa's method

- Take $S = \{0, 1, 3\}$, so that $|S + S| < |S - S|$.
- Fix a probability $0 < q < 1$.
- Choose each $\mathbf{x} \in S^d \subset \mathbb{Z}^d$ independently with probability $q^d$.
- Project this set into $\mathbb{Z}$ using a suitable map $\phi$ of the form

$$\phi(x_1, \ldots, x_d) = \lambda_1 x_1 + \cdots + \lambda_d x_d.$$

# Why does this work?

$$a = (3,1,1,3,3,3,3,3,3,0,1,1,3,3,1)$$
$$b = (0,0,3,0,1,0,0,0,3,3,1,1,0,1,0)$$
$$a+b = (3,1,4,3,4,3,3,3,6,3,2,2,3,4,1)$$
$$a-b = (3,1,\bar{2},3,2,3,3,3,0,\bar{3},0,0,3,2,1)$$

In $a+b$:

- $1,3,4$ can be made in two ways;
- $0,2,6$ can be made in one way.

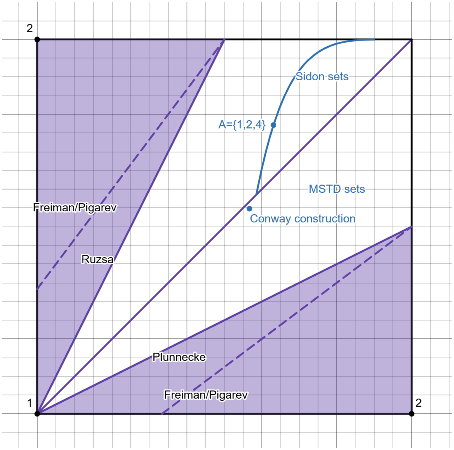
In $a-b$:

- $0$ can be made in three ways,
- but everything else can only be made in one way.

So, for the right choice of $q$, a lot of the sums will coincide, but the differences will spread out.

The trick is to choose the right value of $q$, depending on the *multiplicity spectra* of $S+S$ and $S-S$.

# The feasible region $F_{1,-1}$ and Ruzsa's method




$$E_{1,-1} = \left\{ \left( \frac{\log|A+A|}{\log|A|}, \frac{\log|A-A|}{\log|A|} \right) \right\}, F_{1,-1} = \overline{E_{1,-1}}$$

# Sums of dilates

Recall that the sumset $A + B$ of sets $A, B \subset \mathbb{Z}^d$ is defined as

$$A + B = \{a + b : a \in A, b \in B\}.$$

For a given integer $k$, the dilate $A + k \cdot B$ is defined as

$$A + k \cdot B = \{a + kb : a \in A, b \in B\}.$$

Note: $A + 2 \cdot A$ (for example) is generally a strict subset of $3A = A + A + A$, where each of the three summands can be distinct.

Example: If $A = \{1, 2, 4\}$, then $A + 2 \cdot A = \{3, 4, 5, 6, 8, 9, 10, 12\}$, but $A + A + A = \{3, 4, 5, 6, 7, 8, 9, 10, 12\}$.

# A complicated formula

Suppose

$$|S| = M \qquad |S + k \cdot S| = N \qquad S + k \cdot S = \{x_1, \ldots, x_N\}.$$

For $1 \leq i \leq N$, write $\lambda_i$ for the number of ordered pairs $(s_1, s_2) \in S^2$ such that $x_i = s_1 + ks_2$. $\lambda_1, \lambda_2, \ldots, \lambda_N$ is the *multiplicity spectrum* of $S + k \cdot S$. Note that $\sum_{i=1}^N \lambda_i = M^2$.

**Theorem (CPS)**

Fix a finite set $S \subseteq \mathbb{Z}$, and let $k$ be a nonzero integer. Let $M$, $N$, $\lambda_i$ be defined as above. Then, if $S_n$ is the random set produced by Ruzsa's method, starting with the set $S$, and using the probability $q$, we have

$$\lim_{n\to\infty} (\mathbb{E}|S_n + k \cdot S_n|)^{1/n} = \begin{cases} (qM)^2 & \text{if } q^2 \leq \prod_i \lambda_i^{-\lambda_i/M^2} \\ N & \text{if } q^2 \geq \prod_i \lambda_i^{-1/N} \\ q^{2p} \sum_i \lambda_i^p & \text{if } q^2 = \prod_i \lambda_i^{-\Lambda(p)} \end{cases}$$

where

$$\Lambda(p) = \frac{\lambda_i^p}{\sum_1^N \lambda_j^p}.$$

# The size of a fractional dilate

**Definition (CPS)**

A *fractional dilate* $\alpha$ is a map $\alpha : \mathbb{Z} \to \mathbb{R}^+ \cup \{0\}$ with finite support $\mathrm{supp}(\alpha)$. The *size of a fractional dilate* is

$$\|\alpha\| = \inf_{0 \le p \le 1} \sum_{n \in \mathrm{supp}(\alpha)} \alpha(n)^p.$$

• If $\alpha$ is a *fractional set* ($\alpha(n) \le 1$ for all $n$), then $\|\alpha\| = \sum_{n \in \mathbb{Z}} \alpha(n)$. (Take $p = 1$.)

• If $\alpha(n) \ge 1$ for all $n$, then $\|\alpha\| = |\mathrm{supp}(\alpha)|$. (Take $p = 0$.)

• In general, it is not obvious which value of $p$ to take. If the infimum above is attained at $p = 1$, we say that $\alpha$ is *spartan*. If the infimum is attained at $p = 0$, we say that $\alpha$ is *opulent*. Otherwise $\alpha$ is *p-comfortable*.

# Fractional dilates as generalizations of dilates

**Definition (CPS)**

For fractional sets $\alpha, \beta$ and an integer $k$, $\alpha + k \cdot \beta$ is the fractional dilate

$$(\alpha + k \cdot \beta)(n) = \sum_{\substack{(i,j) \\ i+kj=n}} \alpha(i)\beta(j).$$

- If $\alpha = \mathbb{1}_A$ and $\beta = \mathbb{1}_B$, then $\mathrm{supp}(\alpha + k \cdot \beta) = A + k \cdot B$, and the values $(\alpha + k \cdot \beta)(n)$ record multiplicities.

- If $\alpha = q\mathbb{1}_A$ and $\beta = q\mathbb{1}_B$, then $\mathrm{supp}(\alpha + k \cdot \beta) = A + k \cdot B$, and the values $(\alpha + k \cdot \beta)(n)$ record multiplicities, multiplied by $q^2$.

# Ruzsa's method generalized, and quantified

**Definition (CPS)**

Given a fractional set $\alpha$, a random set $S_n \subseteq \mathbb{Z}^n$ is *drawn from* $\alpha^n$ if each element $(i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n$ is chosen independently with probability

$$\alpha(i_1)\alpha(i_2)\cdots\alpha(i_n).$$

Ruzsa: $\alpha = q\mathbb{1}_{\{0,1,3\}}$

**Definition (CPS)**

Let $\alpha$ be a fractional set with $\|\alpha\| > 1$, and suppose $S_n \subseteq \mathbb{Z}^n$ is drawn from $\alpha^n$. Then

$$\mathbb{E}|S_n| = \|\alpha\|^n \qquad \mathrm{Var}|S_n| \leq \|\alpha\|^n$$

and

$$\lim_{n\to\infty} (\mathbb{E}|S_n + k \cdot S_n|)^{1/n} = \|\alpha + k \cdot \alpha\|.$$

# The Convex Connection I

**Difference Body Inequality**

If $A \subset \mathbb{R}^d$ is convex, $V$ denotes volume, and

$$A - A = \{a - b : a, b \in A\},$$

then

$$2^d V(A) \leq V(A - A) \leq \binom{2d}{d} V(A).$$

- Brunn-Minkowski inequality; equality iff $A$ is centrally symmetric.
- *Rogers-Shephard inequality* (1957); equality iff $A$ is a simplex.

# The Convex Connection II

**Hennecart, Robert, Yudin Construction 1999**

If

$$A_{k,d} = \{(x_1, \ldots, x_{d+1}) : x_1, \ldots, x_{d+1} \geq 0,\ x_1 + \cdots + x_{d+1} = k\},$$

then

$$|A| = \binom{k+d}{d} \qquad |A+A| = \binom{2k+d}{d} \qquad |A-A| = \sum_{t=0}^{\min(d,k)} \binom{d}{t}^2 \binom{k+d-t}{d}.$$

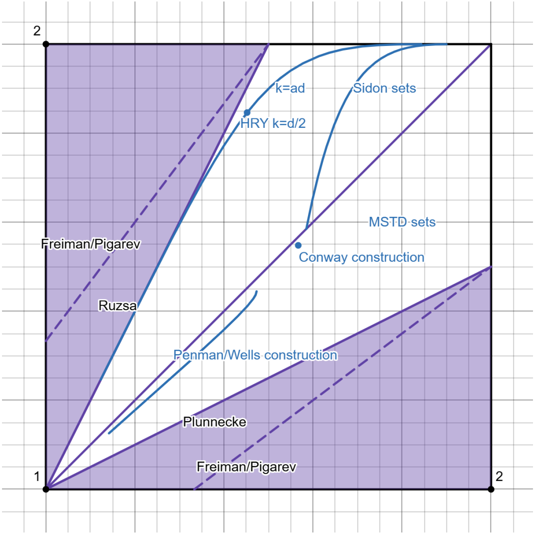
If $k = d/2$, then

$$\frac{\log|A+A|}{\log|A|} \to \frac{4\log 2}{3\log 3 - 2\log 2} \approx 1.4520$$

and

$$\frac{\log|A-A|}{\log|A|} \to \frac{4\log(1+\sqrt{2})}{3\log 3 - 2\log 2} \approx 1.8463,$$

so that $(1.4520, 1.8463)$ is feasible when $k = 1$ and $l = -1$.




$$E_{1,-1} = \left\{ \left( \frac{\log|A+A|}{\log|A|}, \frac{\log|A-A|}{\log|A|} \right) \right\}$$

## Theorem (CPS)

There exists a fractional set $\alpha$ for which

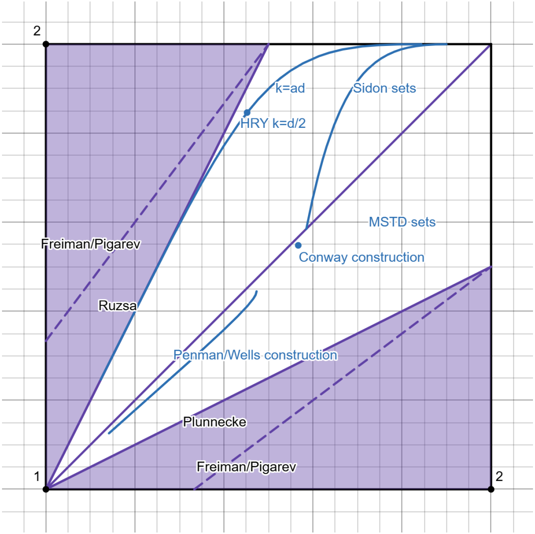
$$\|\alpha\| > 1, \quad \|\alpha - \alpha\| = \|\alpha\|^2, \quad \|\alpha + \alpha\| \leq \|\alpha\|^{1.7354}.$$

## Theorem (CPS)

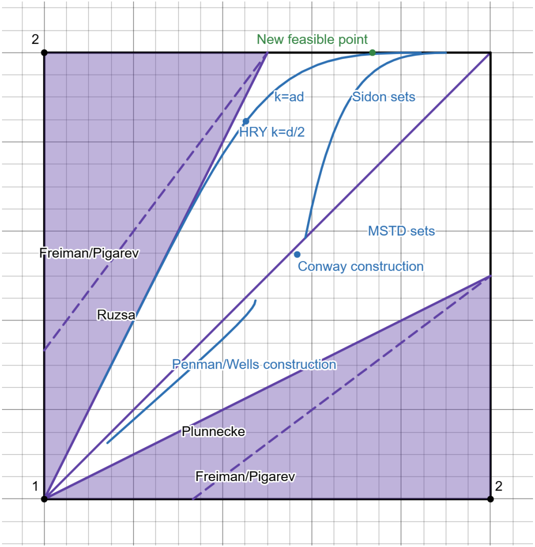
For all $\epsilon > 0$, there exists a finite $A \subset \mathbb{Z}$ such that

$$|A - A| \geq |A|^{2-\epsilon} > 1 \quad \text{and} \quad |A + A| \leq |A|^{1.7354+\epsilon}.$$

So $(1.7354, 2)$ is feasible for $F_{1,-1}$.




$$E_{1,-1} = \left\{ \left( \frac{\log|A+A|}{\log|A|}, \frac{\log|A-A|}{\log|A|} \right) \right\}$$




$$E_{1,-1} = \left\{ \left( \frac{\log|A+A|}{\log|A|}, \frac{\log|A-A|}{\log|A|} \right) \right\}$$

# Sums and dilates

Recall that the sumset $A + A$ of a set $A \subset \mathbb{Z}^d$ is defined as

$$A + A = \{a + a' : a, a' \in A\}.$$

For a given integer $k$, the dilate $A + k \cdot A$ is defined as

$$A + k \cdot A = \{a + ka' : a, a' \in A\}.$$

Note: $A + 2 \cdot A$ (for example) is generally a strict subset of $3A = A + A + A$, where each of the three summands can be distinct.

Example: If $A = \{1, 2, 4\}$, then $A + 2 \cdot A = \{3, 4, 5, 6, 8, 9, 10, 12\}$, but $A + A + A = \{3, 4, 5, 6, 7, 8, 9, 10, 12\}$.

# A question of Bukh

Suppose $A \subset \mathbb{Z}$ has small doubling, so that $|A+A| \leq K|A|$, where $K$ is small. Then, from Plünnecke's inequality,

$$|A + 2 \cdot A| \leq |A + A + A| \leq K^3|A|.$$

Bukh (2008): can the exponent 3 be improved?

**Theorem (Hanson and Petridis 2021)**

If $A \subset \mathbb{Z}$ and $|A+A| \leq K|A|$, then

$$|A + 2 \cdot A| \leq K^{2.95}|A|.$$

**Theorem (Hanson and Petridis 2021)**

If $A \subset \mathbb{Z}$ and $|A+A| \leq K|A|$, then

$$|A + 2 \cdot A| \leq (K|A|)^{4/3}.$$

How do we make $A + 2 \cdot A$ large, while keeping $A + A$ small?

# Consider the hypercube

How do we make $A + 2 \cdot A$ large, while keeping $A + A$ small?

Consider the hypercube:

$$H_n = \left\{ \sum_{i=0}^{n-1} a_i 4^i : \; a_i \in \{0, 1\} \right\},$$

that is, all integers from 0 to $\frac{1}{3}(4^n - 1)$ with only 0s and 1s in their base-4 expansions. We have

$$|H_n| = 2^n \qquad |H_n + H_n| = 3^n \qquad |H_n + 2 \cdot H_n| = 4^n.$$

How good is this? For all $n$,

$$\left( \frac{\log |H_n + H_n|}{\log |H_n|}, \frac{\log |H_n + 2 \cdot H_n|}{\log |H_n|} \right) = \left( \frac{\log 3}{\log 2}, 2 \right) \approx (1.585, 2).$$

# A modified hypercube

How do we make $A + 2 \cdot A$ large, while keeping $A + A$ small?

Fix $\alpha \in \left(\frac{1}{2}, 1\right)$, and set $k = \lfloor \alpha n \rfloor$.

Consider the hypercube + interval $A_{n,k} = H_n \cup I_k$, where

$$H_n = \left\{ \sum_{i=0}^{n-1} a_i 4^i : \; a_i \in \{0,1\} \right\} \qquad I_k = \left\{ 0, 1, \dots, \frac{4(4^{k-1} - 1)}{3} \right\}.$$

As $n \to \infty$,

$$\left( \frac{\log |A_{n,k} + A_{n,k}|}{\log |A_{n,k}|}, \frac{\log |A_{n,k} + 2 \cdot A_{n,k}|}{\log |A_{n,k}|} \right) \to \left( \max \left\{ \frac{\log 3}{\alpha \log 4}, \frac{1+\alpha}{2\alpha} \right\}, \frac{1}{\alpha} \right).$$

How does this compare with the upper bounds of Hanson and Petridis?

# Feasible regions in general

How does this compare with the upper bounds of Hanson and Petridis?

## Definition

For fixed integers $k$ and $l$, we define the *feasible region* $F_{k,l}$ *to be the closure of the set* $E_{k,l}$ *of attainable points*

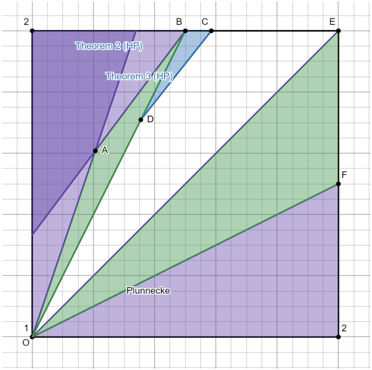
$$E_{k,l} = \{(x,y) : |A + k \cdot A| = |A|^x, |A + l \cdot A| = |A|^y \text{ for some } A \subset \mathbb{Z}\}$$

*as* $A$ *ranges over finite sets of integers.* $F_{k,l} \subset [1,2]^2$.

## Proposition

For all nonzero $k, l$, the feasible region $F_{k,l}$ is convex, and contains the diagonal $D = \{(x,x) : 1 \leq x \leq 2\}$.

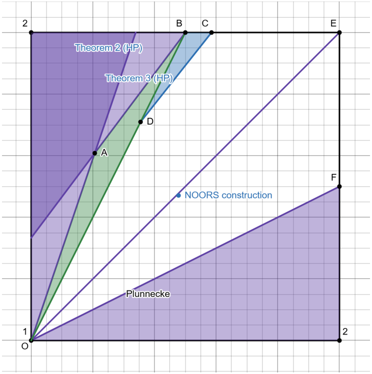
# The feasible region $F_{1,2}$




$$E_{1,2} = \left\{ \left( \frac{\log |A+A|}{\log |A|}, \frac{\log |A+2\cdot A|}{\log |A|} \right) \right\}, F_{1,2} = \overline{E_{1,2}}$$

Purple = forbidden (by Plünnecke, Hanson and Petridis)
White = known to be feasible (ODC = hypercube + interval)
Green + blue = unknown

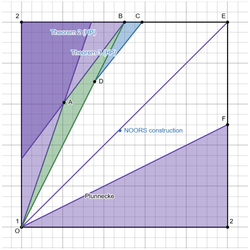
# Surprise




$$E_{1,2} = \left\{ \left( \frac{\log |A+A|}{\log |A|}, \frac{\log |A+2\cdot A|}{\log |A|} \right) \right\}, F_{1,2} = \overline{E_{1,2}}$$

Purple = forbidden (by Plünnecke, Hanson and Petridis)
White = known to be feasible (ODC = hypercube + interval)
Green + blue = unknown

# Various constructions




$$E_{1,2} = \left\{ \left( \frac{\log|A+A|}{\log|A|}, \frac{\log|A+2\cdot A|}{\log|A|} \right) \right\}, F_{1,2} = \overline{E_{1,2}}$$

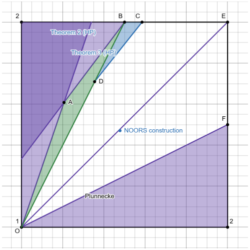
**Nathanson, O'Bryant, Orosz, Ruzsa and Silva (2007)**

…constructed a $A$ set of size 2646 with $|A + 2 \cdot A| < |A + A|$.

**Jeck Lim (2025)**

Simpler construction

# A large set

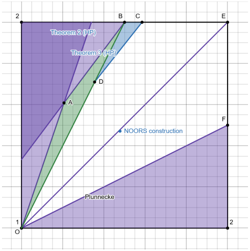



$$E_{1,2} = \left\{ \left( \frac{\log|A+A|}{\log|A|}, \frac{\log|A+2\cdot A|}{\log|A|} \right) \right\}, F_{1,2} = \overline{E_{1,2}}$$

**Nathanson, O'Bryant, Orosz, Ruzsa, Silva construction**

$$\begin{aligned} R_{13} &= \{0,1,6,7,9,11\} \\ R_{15} &= \{0,1,5,6,10,11,13\} \\ R_{16} &= \{0,1,3,5,7,9,11,13,15\} \\ R_{19} &= \{0,1,11,12,14,16,18\} \\ A &= \{x \in [1, 13\cdot 15\cdot 16\cdot 19] : x \equiv i_m \pmod{m} \text{ with } i_m \in R_m \text{ for } m = 13,15,16,19\} \end{aligned}$$

# Jeck Lim's construction




$$E_{1,2} = \left\{ \left( \frac{\log |A+A|}{\log |A|}, \frac{\log |A+2\cdot A|}{\log |A|} \right) \right\}, F_{1,2} = \overline{E_{1,2}}$$

**Jeck Lim's construction**

We first start with a certain construction mod p which is not hard to find: there is a prime p and a set B in Z/p such that B+B=Z/p but B+2.B does not contain 0 or 1. Now let k be a large integer, and consider A to be the set of all k-digit numbers written in base p, whose digits are in B. Then A+A definitely contains all k-digit numbers (all in base p), whereas A+2.A contains at most the (k+1)-digit numbers without the digit 1, which is exponentially less.

$$S = \{1, 2, 3, 6, 9, 12, 13, 14\} \subseteq \mathbb{Z}_{22}$$

# Some new constructions

## Construction minimzing $|A|$

$$S = \{0, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 20\} \subseteq \mathbb{Z}_{40}$$

When $k = 3$, we have

$$|A| = 1728 \qquad |A + A| = 65641 \qquad |A + 2 \cdot A| = 65017$$

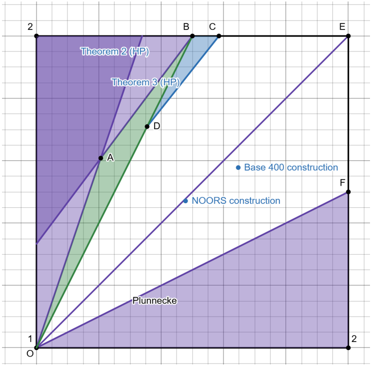
## Construction minimizing $\log |A + 2 \cdot A| / \log |A + A|$

$$S = \{0, 1, 2, \ldots, 9, 19, 29, \ldots, 199, 200, 201, 202, \ldots, 208\} \subseteq \mathbb{Z}_{400}$$

Asymptotically,

$$|A| \sim 38^k \quad |A+A| \sim 400^k \quad |A+2\cdot A| \sim (160+6\sqrt{631})^k = (310.7\cdots)^k$$

# The feasible region $F_{1,2}$




$$E_{1,2} = \left\{ \left( \frac{\log |A+A|}{\log |A|}, \frac{\log |A+2\cdot A|}{\log |A|} \right) \right\}, F_{1,2} = \overline{E_{1,2}}$$

Purple = forbidden (by Plünnecke, Hanson and Petridis)

White = known to be feasible (ODC = hypercube + interval)

Green + blue = unknown

# Open problems

- Determine the feasible regions $F_{1,-1}$ and $F_{1,2}$.

- Find another use for fractional dilates.

Thank you for your attention!