Sums, Differences and Dilates

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Recall that the sumset $A + A$ of a set $A \subset \mathbb{Z}^d$ is defined as

$$A + A = \{a + a' : a, a' \in A\}.$$

For a given integer k, the dilate $A + k \cdot A$ is defined as

$$A + k \cdot A = \{a + ka' : a, a' \in A\}.$$

Note: $A + 2 \cdot A$ (for example) is generally a strict subset of $3A = A + A + A$, where each of the three summands can be distinct.

Example: If $A = \{1, 2, 4\}$, then $A + 2 \cdot A = \{3, 4, 5, 6, 8, 9, 10, 12\}$, but $A + A + A = \{3, 4, 5, 6, 7, 8, 9, 10, 12\}$.

Suppose $A \subset \mathbb{Z}$ has small doubling, so that $|A + A| \leq K|A|$, where K is small. Then, from Plünnecke's inequality,

$$|A + 2 \cdot A| \leq |A + A + A| \leq K^3|A|.$$

Bukh (2008): can the exponent 3 be improved?

Theorem (Hanson and Petridis 2021)

If $A \subset \mathbb{Z}$ and $|A + A| \leq K|A|$, then

$$|A + 2 \cdot A| \leq K^{2.95}|A|.$$

Theorem (Hanson and Petridis 2021)

If $A \subset \mathbb{Z}$ and $|A + A| \leq K|A|$, then

$$|A + 2 \cdot A| \leq (K|A|)^{4/3}.$$

How do we make $A + 2 \cdot A$ large, while keeping $A + A$ small?

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Consider the hypercube:

$$H_n = \left\{ \sum_{i=0}^{n-1} a_i 4^i : \; a_i \in \{0, 1\} \right\},$$

that is, all integers from 0 to $\frac{1}{3}(4^n - 1)$ with only 0s and 1s in their base-4 expansions. We have

$$|H_n| = 2^n \qquad |H_n + H_n| = 3^n \qquad |H_n + 2 \cdot H_n| = 4^n.$$

How good is this? For all n,

$$\left(\frac{\log |H_n + H_n|}{\log |H_n|}, \frac{\log |H_n + 2 \cdot H_n|}{\log |H_n|} \right) = \left(\frac{\log 3}{\log 2}, 2 \right) \approx (1.585, 2).$$

How do we make $A + 2 \cdot A$ large, while keeping $A + A$ small?

Fix $\alpha \in \left(\frac{1}{2}, 1\right)$, and set $k = \lfloor \alpha n \rfloor$.

Consider the hypercube + interval $A_{n,k} = H_n \cup I_k$, where

$$H_n = \left\{ \sum_{i=0}^{n-1} a_i 4^i : \ a_i \in \{0,1\} \right\} \qquad I_k = \left\{ 0, 1, \ldots, \frac{4(4^{k-1}-1)}{3} \right\}.$$

As $n \to \infty$,

$$\left(\frac{\log |A_{n,k} + A_{n,k}|}{\log |A_{n,k}|}, \frac{\log |A_{n,k} + 2 \cdot A_{n,k}|}{\log |A_{n,k}|} \right) \to \left(\max \left\{ \frac{\log 3}{\alpha \log 4}, \frac{1+\alpha}{2\alpha} \right\}, \frac{1}{\alpha} \right).$$

How does this compare with the upper bounds of Hanson and Petridis?

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Definition

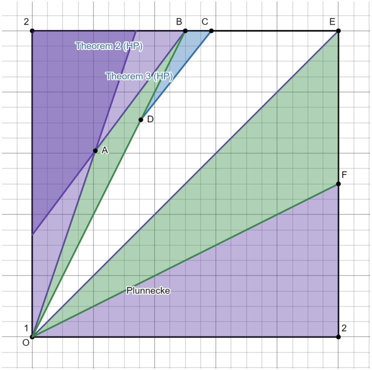
For fixed integers k and l, we define the *feasible region* $F_{k,l}$ *to be the closure of the set* $E_{k,l}$ *of attainable points*

$$E_{k,l} = \left\{ \left(\frac{\log|A + k \cdot A|}{\log|A|}, \frac{\log|A + l \cdot A|}{\log|A|} \right) \right\},$$

as A *ranges over finite sets of integers.* $F_{k,l} \subset [1,2]^2$*.*

Proposition

For all nonzero k, l, the feasible region $F_{k,l}$ is convex, and contains the diagonal $D = \{(x,x) : 1 \leq x \leq 2\}$.

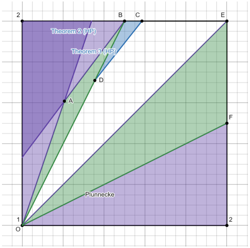


$$E_{1,2} = \left\{ \left(\frac{\log |A+A|}{\log |A|}, \frac{\log |A+2\cdot A|}{\log |A|} \right) \right\}$$

Purple = forbidden (by Plünnecke, Hanson and Petridis)

White = known to be feasible (ODC = hypercube + interval)

Green + blue = unknown (“triangles of sadness”)



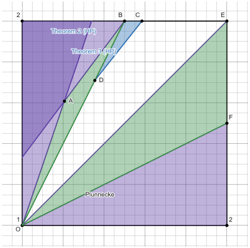
$$E_{1,2} = \left\{ \left(\frac{\log|A+A|}{\log|A|}, \frac{\log|A+2\cdot A|}{\log|A|} \right) \right\}$$

Definition

A *Sidon set* A contains no nontrivial $(a, b, c, d) \in A^4$ with $a + b = c + d$.

Theorem

If A is a Sidon set with at least two elements, then $|A + 2 \cdot A| > |A + A|$.



$$E_{1,2} = \left\{ \left(\frac{\log|A+A|}{\log|A|}, \frac{\log|A+2\cdot A|}{\log|A|} \right) \right\}$$

Theorem (1970s)

Suppose A and B are subsets of the hypercube $\{0,1\}^n$. Then

$$|A+2\cdot B| = |A||B| \leq |A+B|^p,$$

where $p = \log 4/\log 3$.

Line CE is *precisely* the feasible line in $F_{1,2}$ for subsets of $\{0,1\}^n$.

How about differences, i.e., $A - A$ vs $A + A$?

Theorem (Conway 1967)

If $A = \{0, 2, 3, 4, 7, 11, 12, 14\}$, then $|A + A| = 26 > 25 = |A - A|$.

Theorem (Freiman and Pigarev 1973)

$$|A + A|^{3/4} \leq |A - A| \leq |A + A|^{4/3}$$

Theorem (Ruzsa 1976, 1989)

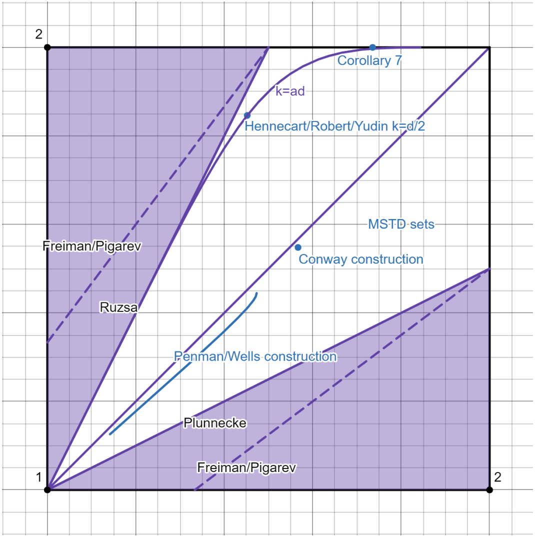
$$\left(\frac{|A + A|}{|A|}\right)^{\frac{1}{2}} \leq \frac{|A - A|}{|A|} \leq \left(\frac{|A + A|}{|A|}\right)^{2}$$

The upper bound comes from taking $B = C = -A$ in Ruzsa's triangle inequality:

$$|A||B - C| \leq |A - B||A - C|,$$

and the lower bound comes from taking $B = C = -A$ in this inequality:

$$|A||B + C| \leq |A + B||A + C|.$$



$$E_{1,-1} = \left\{ \left(\frac{\log |A+A|}{\log |A|}, \frac{\log |A-A|}{\log |A|} \right) \right\}$$

Observation

If $A \subset \mathbb{Z}$ with $|A| = n$, then

$$|A + A| \leq \frac{n(n+1)}{2} \qquad \text{and} \qquad |A - A| \leq n^2 - n + 1,$$

with equality in both cases precisely when A is a Sidon set.

Theorem (Ruzsa 1992)

For every large enough n, there is a set A with $|A| = n$,

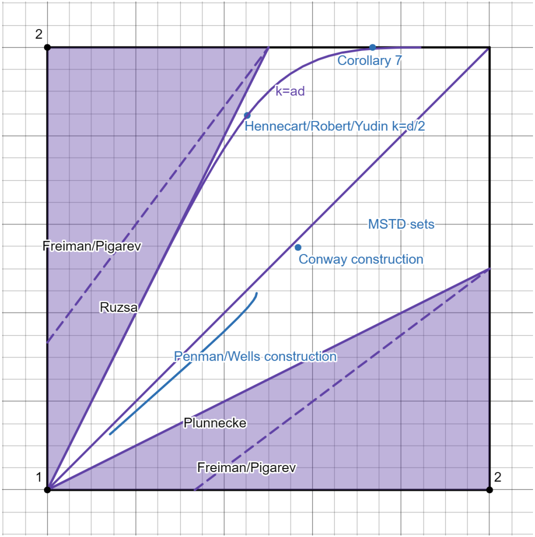
$$|A + A| \leq n^{2-c} \qquad \text{and} \qquad |A - A| \geq n^2 - n^{2-c},$$

where c is a positive absolute constant, and a set B with $|B| = n$,

$$|B - B| \leq n^{2-c'} \qquad \text{and} \qquad |B + B| \geq \tfrac{1}{2}n^2 - n^{2-c'},$$

where c' is a positive absolute constant.

So $(2 - c, 2)$ and $(2, 2 - c')$ are both feasible when $k = 1$ and $l = -1$.

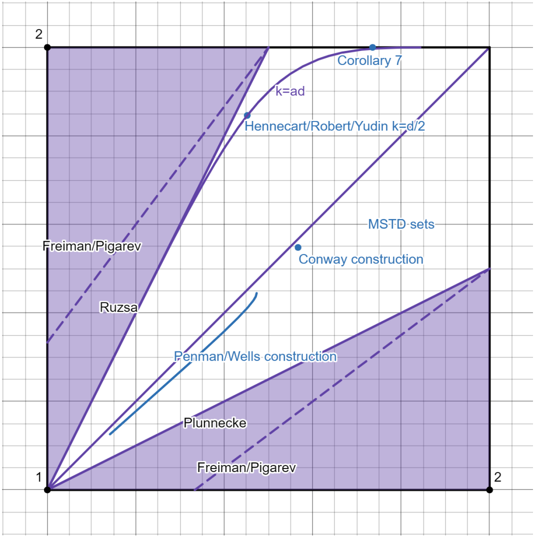


$$E_{1,-1} = \left\{ \left(\frac{\log |A+A|}{\log |A|}, \frac{\log |A-A|}{\log |A|} \right) \right\}$$

Ruzsa's method:

- Take $S = \{0, 1, 3\}$, so that $|S + S| < |S - S|$.
- Fix a probability $0 < q < 1$.
- Choose each $\mathbf{x} \in S^d \subset \mathbb{Z}^d$ independently with probability q^d.
- Project this set into $\mathbb{Z}$ using a suitable map ϕ of the form

$$\phi(x_1, \ldots, x_d) = \lambda_1 x_1 + \cdots + \lambda_d x_d.$$



$$E_{1,-1} = \left\{ \left(\frac{\log |A+A|}{\log |A|}, \frac{\log |A-A|}{\log |A|} \right) \right\}$$

Definition (CPS)

A *fractional dilate* γ is a map $\gamma : \mathbb{Z} \to \mathbb{R}^+ \cup \{0\}$ with finite support $\mathrm{supp}(\gamma)$. The *size of a fractional dilate* is

$$\|\gamma\| = \inf_{0 \le p \le 1} \sum_{n \in \mathrm{supp}(\gamma)} \gamma(n)^p.$$

- If α is a *fractional set* ($\alpha(n) \le 1$ for all n), then $\|\alpha\| = \sum_{n \in \mathbb{Z}} \alpha(n)$.

Definition (CPS)

For fractional sets α, β and an integer k, $\alpha + k \cdot \beta$ is the fractional dilate

$$(\alpha + k \cdot \beta)(n) = \sum_{\substack{(i,j) \\ i+kj=n}} \alpha(i)\beta(j).$$

Definition (CPS)

Given a fractional set α, a random set $S_n \subseteq \mathbb{Z}^n$ is *drawn from* α^n if each element $(i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n$ is chosen independently with probability

$$\alpha(i_1)\alpha(i_2)\cdots\alpha(i_n).$$

Ruzsa: $\alpha = q\mathbb{1}_{\{0,1,3\}}$

Definition (CPS)

Let α be a fractional set with $\|\alpha\| > 1$, and suppose $S_n \subseteq \mathbb{Z}^n$ is drawn from α^n. Then

$$\mathbb{E}|S_n| = \|\alpha\|^n \qquad \mathrm{Var}|S_n| \leq \|\alpha\|^n$$

and

$$\lim_{n\to\infty} (\mathbb{E}|S_n + k \cdot S_n|)^{1/n} = \|\alpha + k \cdot \alpha\|.$$

Difference Body Inequality

If $A \subset \mathbb{R}^d$ is convex, V denotes volume, and

$$A - A = \{a - b : a, b \in A\},$$

then

$$2^d V(A) \leq V(A - A) \leq \binom{2d}{d} V(A).$$

- Lower bound comes from Brunn-Minkowski inequality; equality iff A is centrally symmetric
- Upper bound: *Rogers-Shephard inequality* (1957); equality iff A is a simplex

Hennecart, Robert, Yudin Construction 1999

If

$$A_{k,d} = \{(x_1, \ldots, x_{d+1}) : x_1, \ldots, x_{d+1} \geq 0,\ x_1 + \cdots + x_{d+1} = k\},$$

then

$$|A| = \binom{k+d}{d} \qquad |A+A| = \binom{2k+d}{d} \qquad |A-A| = \sum_{t=0}^{\min(d,k)} \binom{d}{t}^2 \binom{k+d-t}{d}.$$

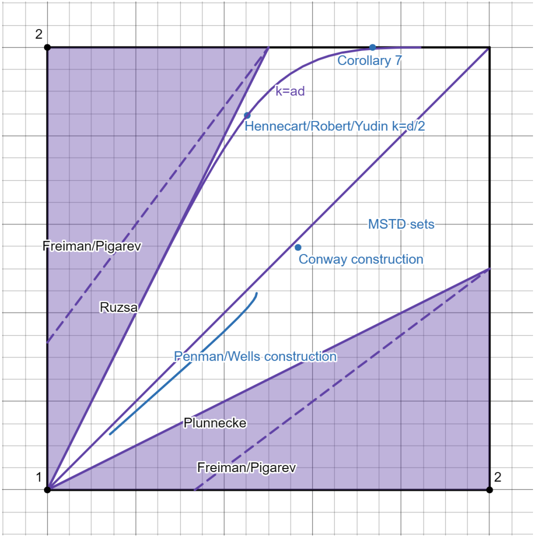
If $k = d/2$, then

$$\frac{\log|A+A|}{\log|A|} \to \frac{4\log 2}{3\log 3 - 2\log 2} \approx 1.4520$$

and

$$\frac{\log|A-A|}{\log|A|} \to \frac{4\log(1+\sqrt{2})}{3\log 3 - 2\log 2} \approx 1.8463,$$

so that $(1.4520, 1.8463)$ is feasible when $k = 1$ and $l = -1$.



$$E_{1,-1} = \left\{ \left(\frac{\log |A+A|}{\log |A|}, \frac{\log |A-A|}{\log |A|} \right) \right\}$$

Theorem (CPS)

There exists a fractional set α for which

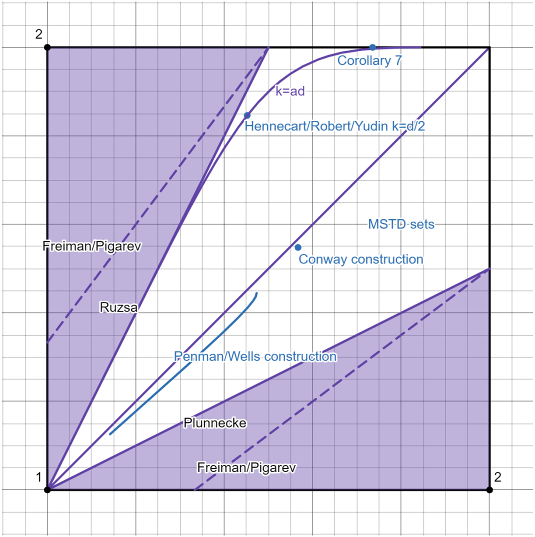
$$\|\alpha\| > 1, \quad \|\alpha - \alpha\| = \|\alpha\|^2, \quad \|\alpha + \alpha\| \leq \|\alpha\|^{1.7354}.$$

Theorem (CPS)

For all $\epsilon > 0$, there exists a finite $A \subset \mathbb{Z}$ such that

$$|A - A| \geq |A|^{2-\epsilon} > 1 \quad \text{and} \quad |A + A| \leq |A|^{1.7354+\epsilon}.$$

So $(1.7354, 2)$ is feasible for $F_{1,-1}$.



$$E_{1,-1} = \left\{ \left(\frac{\log|A+A|}{\log|A|}, \frac{\log|A-A|}{\log|A|} \right) \right\}$$

Two open problems:

- Is there is a $A \subset \mathbb{Z}$ with $|A + A| > |A + 2 \cdot A|$?

- Is there another use for fractional dilates?