Gallai's Theorem

Let $V = \{v_1, \ldots, v_t\} \in \mathbb{R}^d$ be a finite set of points. A set $W = \{w_1, \ldots, w_t\} \in \mathbb{R}^d$ is said to be *homothetic* to V, if, after possibly rearranging the w_i, there exists a nonzero $c \in \mathbb{R}$ and also $b \in \mathbb{R}^d$ such that

$$w_i = cv_i + b$$

for each i. In other words, W is a scaled and translated (but not rotated) copy of V.

Gallai's theorem states the following

Theorem 1. *Let $V \in \mathbb{R}^d$ be finite. Then any finite coloring of $\mathbb{R}^d$ contains a monochromatic W homothetic to V.*

Proof. This is often called a "one line" deduction from the Hales-Jewett theorem, but it is a slightly tricky line to reconstruct!

Let $V = \{v_1, \ldots, v_t\} \in \mathbb{R}^d$, and let χ be an r-coloring of $\mathbb{R}^d$. We're going to apply the Hales-Jewett theorem, so we need the monochromatic line it gives us to yield a monochromatic W. Since $|W| = t$, it makes sense to apply the Hales-Jewett theorem with r colors and cube side-length t. Accordingly, let $n = HJ(r, t)$. So far, so good.

Next, we need to use χ to define an r-coloring χ' of the cube $[t]^n = \{1, \ldots, t\}^n$ which does the job. Here it is (this is the "one line"):

$$\chi'(a_1, \ldots, a_n) = \chi\left(\sum_{j=1}^{n} v_{a_j}\right) = \chi\left(\sum_{i=1}^{t} n_i v_i\right),$$

where, for $1 \leq i \leq t$, $n_i = n_i(a_1, \ldots, a_n)$ is the number of times i appears in $(a_1, \ldots, a_n)$.

Why does this work? First, we apply the Hales-Jewett theorem to get a monochromatic line $L \subseteq [t]^n$ (perhaps this is the "one line" - who knows). L has active coordinates from an index set $I \subseteq [n]$, and fixed coordinates f_i for $i \in F = [n] \setminus I$. The fact that L is monochromatic under χ' means that, under χ, all the points

$$\sum_{i \in F} v_{f_i} + |I| v_i$$

receive the same color. But these points comprise our homothetic copy of V, with

$$b = \sum_{i \in F} v_{f_i} \quad \text{and} \quad c = |I| \neq 0.$$

$\square$

A couple of remarks are in order. First, the dimension d does not really feature in the proof. Second, given χ and V, the proof tells us to examine only the colors of a finite set of points in $S_V \subseteq \mathbb{R}^d$. Let's look at a few examples.

• $d = 1, r = 2$ and $V = \{0, 1\}$

Let $n = HJ(2,2) = 2$. (It is not hard to check that $HJ(2,2) = 2$: two of the points $(1,1), (1,2)$ and $(2,2)$ must receive the same color, and these points form a monochromatic line.) The proof tells us that among the points $\{0, 1, 2\}$ we can find a homothetic copy of V. Since a homothetic copy of V is just two identically-colored points, we could have figured this out without the Hales-Jewett theorem.

• $d = 1, r = 2$ and $V = \{0, 1, 2\}$

Let $n = HJ(2,3) = 4$. (This is non-trivial! It was established in [1].) This time, the proof tells us that among the points $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ we can find a homothetic copy of V. Now a homothetic copy of V is just a three-term arithmetic progression, so we've proved that the van der Waerden number $W(3,2)$ satisfies $W(3,2) \leq 9$. This is much better than our earlier bound $W(3,2) \leq 325$, and, since in fact $W(3,2) = 9$, it is actually the best bound we could have hoped for.

• $d = 2, r = 2$ and $V = \{\mathbf{0}, \mathbf{i}, \mathbf{i}'\}$, where $\mathbf{i}' = (\frac{1}{2}, \frac{\sqrt{3}}{2})$

We're looking for monochromatic equilateral triangles with a horizontal base (either the "right way up" or "upside down", although the proof above supplies a triangle that is the "right way up"). The proof tells us to examine just the 15 points $a\mathbf{i} + b\mathbf{i}'$, where $a, b \in \mathbb{Z}$ with $a, b \geq 0$ and $a + b \leq 4$. And indeed it is easy to check that any 2-coloring of these 15 points contains a homothetic copy W of V. This is "best possible", in the sense that if we omit the point $4\mathbf{i}$, there's a 2-coloring of the remaining 14 points with no such W.

References

[1] Hindman, N. and Tressler, E., The first nontrivial Hales-Jewett number is four, *Ars Combinatoria* **113** (2014), 385–390.