Folkman's Theorem

For real numbers $x_1, \ldots, x_n$, we define $\mathrm{FS}(x_1, \ldots, x_n)$ to be the set of (nonempty) *finite sums* of the x_i, so that

$$\mathrm{FS}(x_1, \ldots, x_n) = \left\{ \sum_{i \in I} x_i : \emptyset \neq I \subset [n] \right\}.$$

Folkman's theorem states the following

Theorem 1. *For all positive integers k and r, there exists a least integer $F(k,r)$ such that any r-coloring of $[F(k,r)]$ contains $a_1 < a_2 < \cdots < a_k$ with $\sum_i a_i \leq F(k,r)$ and $\mathrm{FS}(a_1, \ldots, a_k)$ monochromatic.*

To prove the theorem, we require a lemma.

Lemma 2. *For all positive integers k and r, there exists a least integer $G(k,r)$ such that any r-coloring of $[G(k,r)]$ contains $b_1 < b_2 < \cdots < b_k$ with $\sum_i b_i \leq G(k,r)$ and in which the color of $\sum_{i \in I} b_i$ depends only on $\max I$.*

Proof. We use induction on k and van der Waerden's theorem. Suppose we know that $n = G(k-1, r)$ is finite. I claim that

$$G(k,r) \leq 2W(G(k-1,r)+1, r) = 2W(n+1, r),$$

where $W(x,y)$ denotes the van der Waerden number. To prove this, suppose we are given an r-coloring of $[N]$, where $N = 2W(n+1,r)$. Inside the second half of the interval, $[N/2+1, N]$, we find a monochromatic arithmetic progression of length $n+1$: $\{a, a+d, \ldots, a+nd\}$. Identifying $\{d, 2d, \ldots, nd\}$ with $[n]$, we apply the induction hypothesis to find integers $b_1 < b_2 < \cdots < b_{k-1}$, all divisible by d, with $\sum_i b_i \leq nd$, and where the color of $\sum_{i \in I} b_i$ depends only on $\max I$. Set $b_k = a$ and we are done. □

From here, the rest is easy. Indeed, I claim that

$$F(k, r) \leq G(r(k-1)+1, r).$$

This is because, given an r-coloring of $[N] = [G(r(k-1)+1, r)]$, we use the lemma to find a set $b_1 < b_2 < \cdots < b_{r(k-1)+1}$ in which the color of $\sum_{i \in I} b_i$ depends only on $\max I$. By the pigeonhole principle, at least k of the b_i must receive the same color, so that $b_{i(1)} < b_{i(2)} < \cdots < b_{i(k)}$ are all colored blue, say. But then the entire set $\mathrm{FS}(b_{i(1)}, \ldots, b_{i(k)})$ is blue as well.