Combinatorics: Homework Set 2

November 6, 2009

1. Prove that if $|X| = n$, and if $\mathcal{A} \subset \mathcal{P}(X)$ is an *upset* (i.e. $A \in \mathcal{A}, A \subset B$ imply $B \in \mathcal{A}$) while $\mathcal{B} \subset \mathcal{P}(X)$ is a *downset* (i.e. $A \in \mathcal{B}, B \subset A$ imply $B \in \mathcal{B}$) then

$$2^n|\mathcal{A} \cap \mathcal{B}| \leq |\mathcal{A}||\mathcal{B}|.$$

2. For $\mathcal{A} \subset \mathcal{P}(X)$ write $\mathcal{A} - \mathcal{A} = \{A \setminus B : A, B \in \mathcal{A}\}$. Prove that $|\mathcal{A} - \mathcal{A}| \geq |\mathcal{A}|$.
3. Show that Harper's vertex isoperimetric inequality for the cube implies the Kruskal-Katona theorem.
4. Write down a detailed proof that the neighborhood of an initial segment of the simplicial order in the cube Q^n is itself an initial segment of the simplicial order.
5. The grid $G = [k]^2$ is the graph with vertex set $[k] \times [k]$ and edges joining points (x_1, y_1) and (x_2, y_2) if either $x_1 = x_2$ and $|y_1 - y_2| = 1$ or $y_1 = y_2$ and $|x_1 - x_2| = 1$. State and prove a vertex isoperimetric inequality for G. (Hint: Use compressions.)
6. Show that every 10×10 matrix whose entries are $1, 2, \ldots, 100$ in some order has two neighboring entries (in a row or in a column) that differ by at least 10.
7. Can a countably infinite set have an uncountable collection of nonempty subsets such that the intersection of any two of them is finite?
8. Can a countably infinite set have an uncountable collection of *nested* subsets (i.e. an uncountable collection $\mathcal{A}$ such that for $A, B \in \mathcal{A}$, either $A \subset B$ or $B \subset A$)?
9. Construct a 2-coloring of $\mathbb{N}$ containing no infinite monochromatic arithmetic progression.
10. Construct a set $A \subset \{1, 2, \ldots, 100000\}$ of size $|A| = 1000$ which contains no 3-term arithmetic progression.