Combinatorics: Homework Set 1

October 27, 2009

1. Let $\emptyset \neq \mathcal{A} \subset X^{(r)}, r \geq 1$, and suppose that every set $A \in X^{(r)}$ containing a set $B \in \partial^- \mathcal{A}$ belongs to $\mathcal{A}$. Prove that $\mathcal{A} = X^{(r)}$.
2. Let $x_1, \ldots, x_n \in \mathbb{R}$ with $|x_i| \geq 1$ for all i. Show that at most $\binom{n}{\lfloor n/2 \rfloor}$ of the 2^n sums $s(A) = \sum_{i \in A} x_i$ can lie within any fixed interval $[t, t+1) \subset \mathbb{R}$. (By convention $s(\emptyset) = 0$.)
3. Prove that we have equality in Sperner's theorem only if $\mathcal{A} = X^{(s)}$, with $s = \lfloor n/2 \rfloor$ or $s = \lceil n/2 \rceil$. (**Hint.** We need to have equality at every step of the proof presented in class. Consequently, the case where n is even is easy. When n is odd, say $n = 2r+1$, we must have $\mathcal{A} \subset X^{(r)} \cup X^{(r+1)}$. Also, if $A, B \in \mathcal{A}$ with $\{x_1, \ldots, x_r\} = A \subset B = A \cup \{x_{r+1}\}$, then precisely one of A and B is in $\mathcal{A}$.)
4. Determine the cases of equality in the Erdős-Ko-Rado theorem when $n/3 < r < n/2$.
5. Reformulate the proof of the Erdős-Ko-Rado theorem presented in class in terms of probability.
6. Prove the following extension of the Erdős-Ko-Rado theorem. If $1 \leq r < n/2$ and $\mathcal{A} \subset \bigcup_{i=0}^{r} X^{(i)}$ is an intersecting family then $|\mathcal{A}| \leq \sum_{i=1}^{r} \binom{n-1}{i-1}$.
7. Write out the first 25 sets in the colex order on $\mathbb{N}^{(4)}$.
8. What is the 2009^{th} set in the colex order on $\mathbb{N}^{(4)}$?
9. Give an example of a set system $\mathcal{A} \subset X^{(r)}$ and two disjoint sets $U, V \subset X$ with $|U| = |V|$ and $\max V > \max U$ such that $|\partial^- C_{U,V} \mathcal{A}| > |\partial^- \mathcal{A}|$.
10. Deduce the Erdős-Ko-Rado theorem from the Kruskal-Katona theorem. (**Hint.** Suppose that $\mathcal{A} \subset X^{(r)}$ is intersecting with $1 \leq r < n/2$. Consider the set system $\mathcal{B} \subset X^{(n-r)}$ defined by $\mathcal{B} = \{B \subset X : B = X \setminus A \text{ for some } A \in \mathcal{A}\}$. The argument is due to Daykin.)