Galois Theory: Homework Set 2

February 6, 2013

1. Find the three intermediate fields between $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ and $\mathbb{Q}$, and the corresponding subgroups of $\mathrm{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q})$.

2. Show that every extension of degree $d = 2$ is normal. Is this true for any other d?

3. Let f be a polynomial of degree n over $\mathbb{Q}$, and let $E \subset \mathbb{C}$ be its splitting field. Show that $[E : \mathbb{Q}]$ divides $n!$.

4. Show that if $\alpha_1, \ldots, \alpha_k$ are algebraic numbers, then there exist integers $n_1, \ldots, n_k$ such that $\mathbb{Q}(\alpha_1, \ldots, \alpha_k) = \mathbb{Q}(n_1\alpha_1 + \cdots + n_k\alpha_k)$.

5. Let $E, F \subset \mathbb{C}$. Show that if $[E : F]$ is finite then $E \subset E'$, where E' is normal over F.

6. Determine, with proof, the Galois groups of the following extensions:

   (a) $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) : \mathbb{Q}$

   (b) $\mathbb{Q}\left(\sqrt{2 + \sqrt{2}}\right) : \mathbb{Q}$

   (c) $\mathbb{Q}\left(\sqrt{2 + \sqrt{2 + \sqrt{2}}}\right) : \mathbb{Q}$

   (d) $\mathbb{R} : \mathbb{Q}$

7. Compute the Galois group of (the splitting field of) $x^5 - 2$ over $\mathbb{Q}$.