Galois Theory: Homework Set 1

January 21, 2013

1. Show, using Eisenstein's criterion, that $f(x) = x^3 - 3x - 1$ is irreducible over $\mathbb{Q}$. Let α be a root of f in $\mathbb{C}$. Express $\frac{1}{\alpha}$ and $\frac{1}{\alpha+3}$ as linear combinations of 1, α and α^2.
2. Let $K = F(\alpha)$, where α is a root of the irreducible polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Express $\frac{1}{\alpha}$ in terms of α and the coefficients a_i.
3. Show that $x^4 + 1$ is irreducible over $\mathbb{Q}$, but not over $\mathbb{Q}(\sqrt{2})$.
4. Find the minimum monic polynomial for $\alpha = \frac{1}{5}\sqrt{50 - 10\sqrt{5}}$.
5. Is $\mathbb{Q}(\sqrt{2})$ isomorphic to $\mathbb{Q}(\sqrt{3})$?
6. Prove that $\mathbb{R}$ is not a simple extension of $\mathbb{Q}$.
7. Let $E : F$ be a field extension, and let $\alpha \in E$. Show that multiplication by α is a linear transformation of E considered as a vector space over F. When is this linear transformation non-singular?
8. Let $E : F$ be a field extension, and let p be an irreducible polynomial over F. Show that if the degree of p and $[E : F]$ are coprime then p has no zeros in E.
9. Express $\sqrt[3]{28} - 3$ as a square in $\mathbb{Q}(\sqrt[3]{28})$.
10. Let $\beta = \omega\sqrt[3]{2}$, where $\omega = e^{2\pi i/3}$, and let $K = \mathbb{Q}(\beta)$. Prove that -1 cannot be written as a sum of squares in K.
11. Let $a \in \mathbb{R}$. Solve the system below for x and y:

$$x^2 = y + a$$
$$y^2 = x + a.$$

Interpret your answer geometrically.
12. Let $a \in \mathbb{R}$. Solve the system below for x, y and z:

$$x^2 = y + a$$
$$y^2 = z + a$$
$$z^2 = x + a.$$

Note. Problems 9 and 12 are due to Ramanujan.