

MATH 458/558 Syllabus

This course is an introduction to stochastic processes, using the examples of branching processes, random walks, discrete-time Markov chains, and Poisson processes. No familiarity with measure theory is required, but students are expected to have a working knowledge of basic probability, at the level of MATH 341.

Except for the review, both the theorems listed below and their proofs are examinable. Italicized topics are optional: instructors may choose to include them if time permits.

Review of Basic Probability (approx. 1 week)

Events, probability, conditional probability, independence, Bayes' formula. Discrete random variables, mass functions, examples (Bernoulli, binomial, Poisson, geometric), expectation, variance, conditional expectation, independence, sums of independent random variables, indicator functions. Continuous random variables, distribution and density functions, examples (uniform, exponential, normal), expectation, variance.

Generating Functions (approx. 0.5 weeks)

Probability generating functions: definition and examples. Formulas for expectation and variance. Generating function for the sum of independent random variables, random sum formula.

Branching Processes (approx. 0.5 weeks)

Definitions, offspring distribution X , extinction probability e . Generating function for, and mean size of, n^{th} generation. Characterization of extinction probability e as the smallest non-negative root of $G(x) = x$; $e = 1$ if and only if $\mu \leq 1$ (as long as $\mathbb{P}(X = 1) \neq 1$).

Simple Random Walk on \mathbb{Z} (approx. 1 week)

Definitions, binomial coefficient formulas for position at time n . Recurrence for $p = q$, transience for $p \neq q$. Mean return time $\mathbb{E}(T) = \infty$. Gambler's ruin: definitions and formulas for winning probability and expected length of game for $p = q$.

Discrete-time Markov Chains (approx. 5 weeks)

Definitions and basic properties, the transition matrix. Calculation of n -step transition probabilities. Communicating classes, closed classes, absorption, irreducibility. Calculation of hitting probabilities and mean hitting times; survival probabilities for birth and death chains.

Recurrence and transience; equivalence of transience and summability of n -step transition probabilities; equivalence of recurrence and certainty of return. Recurrence as a class property, relation with closed classes. *Simple random walks in dimensions one, two and three.*

Invariant measures, existence and uniqueness (up to scaling) for irreducible recurrent chains. Mean return time, positive recurrence; equivalence of positive recurrence and the existence of an invariant distribution. Convergence to equilibrium for irreducible, positive recurrent, aperiodic chains, and proof by coupling.

Time reversal, detailed balance, reversibility; random walk on a graph.

Poisson Processes (approx. 1 week)

Definition. Relation to Poisson distribution. Memoryless property and inter-arrival times.