Secrecy coverage in two dimensions

Amites Sarkar*

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Abstract

Working in the infinite plane $\mathbb{R}^2$, consider a Poisson process of *black points* with intensity 1, and an independent Poisson process of *red points* with intensity λ. We grow a disc around each black point until it hits the nearest red point, resulting in a random configuration $\mathcal{A}_\lambda$, which is the union of discs centered at the black points. Next, consider a fixed disc of area n in the plane. What is the probability $p_\lambda(n)$ that this disc is covered by $\mathcal{A}_\lambda$? We prove that if $\lambda^3 n \log n = y$, then, for sufficiently large n, $e^{-8\pi^2 y} \le p_\lambda(n) \le e^{-\frac{2}{3}\pi^2 y}$. The proofs reveal a new and surprising phenomenon, that the obstructions to coverage occur on a wide range of scales.

Keywords: Poisson process; coverage

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1 Introduction

Place discs of radius r in $\mathbb{R}^2$ so that their centers form a Poisson process of intensity 1, and let $B_n \subset \mathbb{R}^2$ be a disc of area $n \gg r^2$. What is the probability that B_n is covered by the small discs? This question, inspired by biology [9], has a long history, and many detailed results are known about it [4, 7]. For instance, writing

$$\pi r^2 = \log n + \log\log n + t,$$

*Department of Mathematics, Western Washington University, Bellingham, WA 98225, USA. Email: amites.sarkar@wwu.edu

Svante Janson proved in 1986 [7] that coverage occurs with probability asymptotically $e^{-e^{-t}}$, as $n \to \infty$. One approach to this result [2, 3] uses the fact that the obstructions to coverage are small uncovered regions, which essentially form their own Poisson process, of intensity e^{-t}/n. Although these uncovered regions may be of different shapes, they are all roughly the same size. Mathew Penrose proved an analogous result for connectivity of the underlying graph [10], where the obstructions – isolated vertices – are again of the same size; the generalized coverage process where the disc radii are independent, identically distributed random variables (the so-called *Boolean model*) has also received much attention [6, 8]. Here we study a simple variant of the original problem, in which the disc radii are no longer independent, and where there are many different obstructions of many different sizes.

To define the process, let $\mathcal{P}$ and $\mathcal{P}'$ be independent Poisson processes, of intensities 1 and λ respectively, in $\mathbb{R}^2$. We will call the points of $\mathcal{P}$ *black points* and the points of $\mathcal{P}'$ *red points*. Place an open disc $D(p, r_p)$ of radius r_p around each black point $p \in \mathcal{P}$, where r_p is maximal so that $D(p, r_p) \cap \mathcal{P}' = \emptyset$. In other words, r_p is the distance from the black point p to the nearest red point $p' \in \mathcal{P}'$ to p. p' is almost surely unique, and we will refer to it as the *stopping point* of the disc centered at p, or of p itself. We thus obtain a random set $\mathcal{A}_\lambda \subset \mathbb{R}^2$ which is the union of discs centered at the points of $\mathcal{P}$. Now let $B_n \subset \mathbb{R}^2$ be a fixed disc of area n, write $B_\lambda(n)$ for the event that $B_n \subset \mathcal{A}_\lambda \cup \mathcal{P}'$ (note that $\mathcal{A}_\lambda \cap \mathcal{P}' = \emptyset$ since the discs $D(p, r_p)$ are open), and set $p_\lambda(n) = \mathbb{P}(B_\lambda(n))$. Since adding red points makes coverage less likely, $p_\lambda(n)$ is a non-increasing function of λ, for fixed n. In addition, $p_\lambda(n)$ is non-increasing in n, with λ fixed, because increasing n corresponds to examining the random set $\mathcal{A}_\lambda$ over a larger area.

This model, based on the *secrecy graph* [5], was inspired by the issue of security in wireless networks, and was studied in [11], where it was proved that if $\lambda^3 n \to \infty$ then $p_\lambda(n) \to 0$, while if $\lambda^3 n (\log n)^3 \to 0$ then $p_\lambda(n) \to 1$. In this paper, we prove that the correct indicator of coverage is $f(\lambda, n) = \lambda^3 n \log n$. Specifically, if $\lambda^3 n \log n = y$, then, for sufficiently large n, $e^{-8\pi^2 y} \le p_\lambda(n) \le e^{-\frac{2}{3}\pi^2 y}$. Interestingly, the proofs indicate that there are obstructions on a range of scales; it seems that, close to the coverage threshold, there will be small uncovered regions whose widths range from around 1 to just above $n^{-1/6}$.

Let us note that the problem of determining the *covered volume fraction* of $\mathcal{A}_\lambda$, which can be defined as $f_\lambda = \mathbb{P}(O \in \mathcal{A}_\lambda)$ (where O is the origin), was solved in [11]. The result is that

$$f_\lambda = 1 - \int_0^\infty f(t)e^{-t/\lambda}\,dt,$$

where $f(t)$ is the (currently unknown) probability density function for the volume of the cell containing the origin O in the Voronoi tessellation formed from $\mathcal{P} \cup \{O\}$, where $\mathcal{P}$ is a unit intensity Poisson process in $\mathbb{R}^2$. This is a genuinely different problem from the present one – it is entirely possible that the expected amount of uncovered area in B_n tends to zero, but that the probability that not all of B_n is covered tends to one. Indeed this does occur for certain values of the parameters.

As motivation for our main results, let us briefly state, and sketch the proof of, the result for the one-dimensional version of our problem. Here, we wish to cover an interval I_n of length n with small intervals centered at black points (a Poisson process with intensity 1), which in turn are stopped by red points (a Poisson process with intensity λ). Denoting the probability of such coverage by $p^1_\lambda(n)$, the result is as follows.

Theorem 1. *If $\lambda^2 n = x$, then $p^1_\lambda(n) \to e^{-4x}$ as $n \to \infty$.*

Proof. Let L be an interval of length ℓ between two consecutive red points in I_n. We wish to compute the probability that L is covered. With this in mind, let m be the midpoint of L, let x be the distance of the closest black point to m lying on the left of m, and let y be the distance of the closest black point to m lying on the right of m. Then coverage of L is determined solely by x and y. Indeed, coverage occurs if and only if $x + y < \ell/2$. Consequently, the probability that L is covered is just $\mathbb{P}(\mathrm{Po}(\ell/2) \geq 2) = 1 - e^{-\ell/2}(1 + \ell/2)$. Next, the unconditional probability that the interval between two consecutive red points is covered, obtained by integrating the above probability against the density function of ℓ, is $(1 + 2\lambda)^{-2} \sim 1 - 4\lambda$. Finally, since there are asymptotically $n\lambda \to \infty$ intervals between consecutive red points, and coverage fails independently in each one with probability asymptotically $4\lambda \to 0$, the number of failures is approximately Poisson with mean $4n\lambda^2 = 4x$, and the result follows. □

The above argument reveals that the obstructions to coverage will typically comprise two red points, distance $O(1)$ apart, without black points sufficiently close to their midpoint to ensure coverage of the interval between them. The set of such intervals is roughly four times as large as its subset consisting of consecutive red points with *no* black point between them. In other words, choosing λ to prohibit such pairs of consecutive red points provides a necessary condition for coverage, $\lambda^2 n \to 0$, which is in fact also sufficient, although such an argument gives the wrong constant in the exponent in Theorem 1. One might expect that a similar situation will exist in two dimensions, namely that if $\lambda^3 n = x$, then $p_\lambda(n)$ tends to e^{-cx}, or possibly some other function of x. The likely obstructions might be triples $\{p, q, r\}$ of red points forming a triangle T, whose sides and area are $O(1)$, and which contains no black points in its interior. However, as we shall show, the truth is more complicated.

The (somewhat unorthodox) organization of this paper is as follows. Since the proof of the main theorem is complicated, we begin by showing the weaker result that if $\lambda^3 n \log n = f(\lambda, n) \to \infty$ then $p_\lambda(n) \to 0$, while if $f(\lambda, n) \to 0$ then $p_\lambda(n) \to 1$. This will be accomplished in the next section, using *good configurations*, which are the key to all that follows. The third and final section contains the proof of our main result, Theorem 4. The proof of Theorem 4 proceeds along similar lines to those of Theorems 2 and 3, but with more careful estimates for both the principal term and the error terms in our approximation of $p_\lambda(n)$.

In this paper, C, C' and C'' denote absolute constants which do not depend on n. We write $f(n) \sim g(n)$ if $f(n)/g(n) \to 1$ as $n \to \infty$, $f(n) = o(1)$ means that $f(n) \to 0$ as $n \to \infty$, and $f(n) = O(1)$ means that, for some constant C, $f(n) \leq C$ for all n. Sometimes we will abuse this notation slightly, so that, for instance, "$O(\log n)$ triangles" means $f(n)$ triangles, where $f(n) \leq C \log n$ for some absolute constant C, and $o(1) + n^2$ means $g(n) + n^2$ with $g(n) \to 0$ as $n \to \infty$. The phrase "with high probability" will mean "with probability tending to 1 as $n \to \infty$"; sometimes this is also written "asymptotically almost surely". Also, for all our results, we will have $\lambda = \lambda(n)$, although we will always suppress the dependence on n.

2 Good configurations

We begin by showing that, in two dimensions, the condition $\lambda^3 n \to 0$ is not sufficient to ensure coverage.

Theorem 2. *If $\lambda^3 n \log n \to \infty$, then $p_\lambda(n) \to 0$.*

Proof. Our strategy will be to show that, under the hypothesis, the expected number of *good configurations* (defined below) tends to infinity. A routine application of the second moment method then shows that a good configuration occurs with high probability (probability tending to one). Finally, we show that a good configuration results in an uncovered region of B_n.

First, therefore, we define a good configuration. Such a configuration, illustrated (though not to scale) in Figure 1, consists of an ordered triple (p, q, r) of red points in B_n. p and q must lie at distance t, where $n^{-1/12} < t < 1$. r must lie at distance between $50/t$ and $100/t$ of p, in such a way that the angle rpq is between $\pi/4$ and $3\pi/4$. (The choice of these angles is somewhat arbitrary: all we need is that the angle rpq is bounded away from 0 and π.) Write ℓ_{ij} for the perpendicular bisector of ij, and S for the bi-infinite strip of width $||p - q||$ centered on ℓ_{pq}. For ease of explanation, suppose that the segment pq is horizontal, so that S is vertical, and that r lies above the line through p and q. ℓ_{pr} and ℓ_{qr} intersect the boundary ∂S of S in four points, the highest of which lies at distance at most $h = 110/t$ from pq. Write $R \subset S$ for the rectangle with base pq and height $2h$ (containing all four intersections above), and $R' \subset S$ for its reflection in pq. A good configuration must also have no black points in the rectangular region $R \cup R'$. Note that the area of $R \cup R'$ is 440, so that, conditioned on the locations of p, q and r, the condition on the black points is satisfied with probability e^{-440}. Now, in a good configuration, given the position of p, q is constrained to lie in some annulus centered at p of area $2\pi t\, dt$, with $n^{-1/12} < t < 1$, and then r must lie in a region of area $7500\pi/4t^2$. Consequently, writing X for the number of good configurations in the fixed disc B_n of area n, there exist absolute constants C and C' such that

$$\mathbb{E}(X) \geq C \int_{n^{-1/12}}^{1} \lambda n \cdot \lambda t^{-2} \cdot \lambda t\, dt = C' \lambda^3 n \log n \to \infty.$$

Second, we show that we can apply the second moment method to prove that, with high probability, $X \geq 1$. For this to work, we require an *upper* bound on λ; it will suffice to assume $\lambda^3 n \to 0$. Since $p_\lambda(n)$ is decreasing in λ, if we can prove that $p_\lambda(n) \to 0$ under the more restrictive hypotheses, the full result will follow. Tessellate B_n with squares of side length $n^{1/6}$, and color a square S_i black if both of its "coordinates" are even and if every point of S_i lies at distance at least $n^{1/6}$ from ∂B_n. (Thus, away from the boundary, one out of every four squares is black.) We will only consider the black squares, which we label $S_1, S_2, \ldots, S_N$. Let the *apex* of a good configuration be the

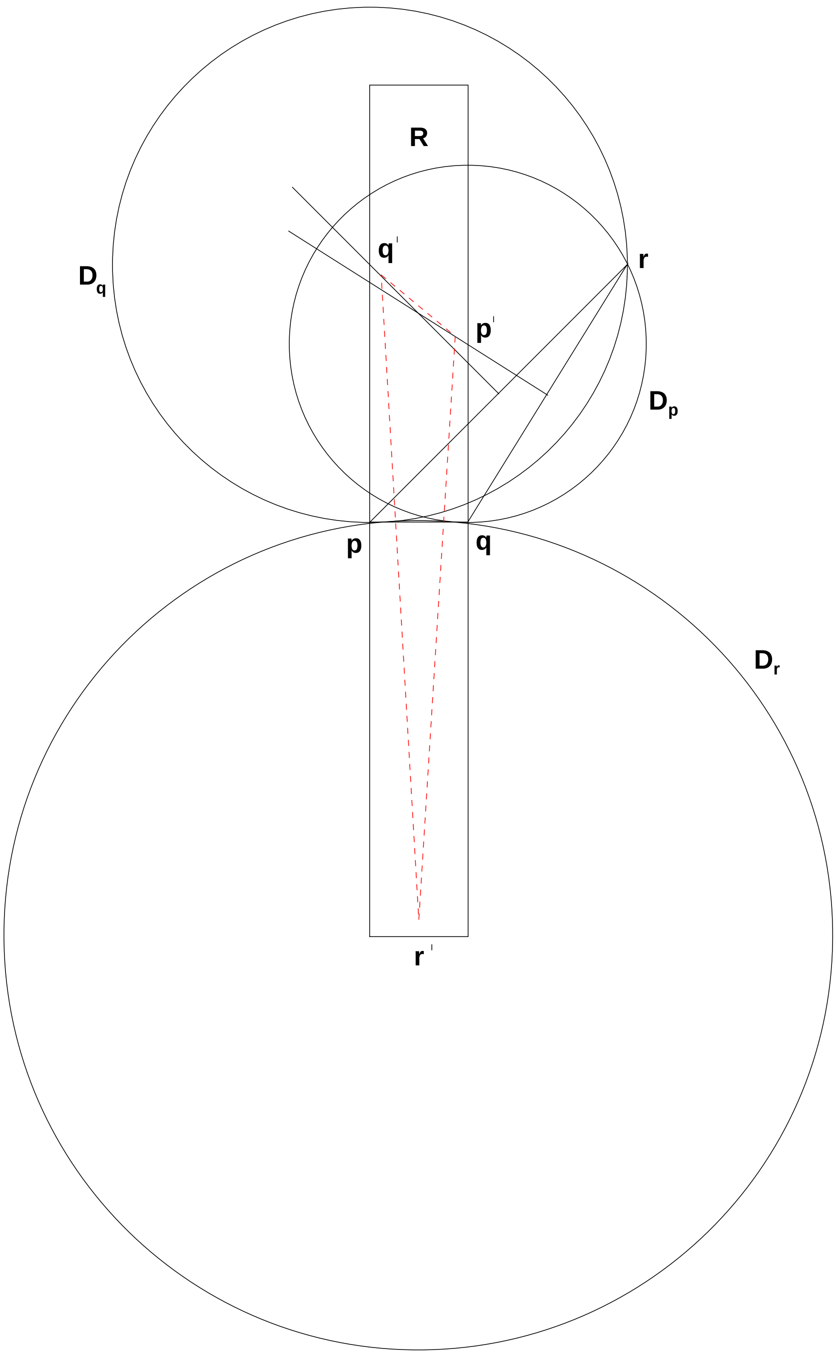

Figure 1: A good configuration. The dashed triangle is the Voronoi cell for the point s' slightly above the midpoint of pq.

point furthest from the opposite side (r, in the above notation), and write X_i for the number of good configurations with apex in S_i. With high probability, each X_i will be either zero or one. Moreover, since the maximum diameter of a good configuration is $O(n^{1/12})$ by construction, the X_i are independent and identically distributed. Let $X' = \sum X_i$. Then $\mathbb{E}(X') \to \infty$ as above, and since, for each i,

$$\mathbb{P}(X_i \geq 1) = O(\log n/n^{2/3}) \to 0, \quad \mathbb{E}(X_i^2) \sim \mathbb{E}(X_i), \quad \mathbb{E}(X_i) \to 0,$$

it follows that

$$\frac{\mathrm{Var}(X')}{\mathbb{E}(X')} = \frac{\mathrm{Var}(X_1)}{\mathbb{E}(X_1)} = \frac{\mathbb{E}(X_1^2) - \mathbb{E}(X_1)^2}{\mathbb{E}(X_1)} \sim \frac{\mathbb{E}(X_1) - \mathbb{E}(X_1)^2}{\mathbb{E}(X_1)} \to 1$$

as $n \to \infty$, and so by Chebyshev's inequality

$$\mathbb{P}(X = 0) \leq \mathbb{P}(X' = 0) \leq \frac{\mathrm{Var}(X')}{\mathbb{E}(X')^2} \sim \frac{1}{\mathbb{E}(X')} \to 0.$$

Finally, we explain why the presence of a good configuration prohibits full coverage. As above, suppose that pq is horizontal, and that r, and hence ℓ_{pr} and ℓ_{qr}, lie above pq. The idea is that part of ℓ_{pq} lying just above pq will be uncovered. Write m_0 for the midpoint of pq, and m_s for the point of ℓ_{pq} at height s above pq. Any black points lying in S above pq and outside R are much closer to r than to p or q, and so their corresponding discs cannot cover m_0 or m_s, for $s \sim C/t$. Write q' for the intersection of ℓ_{pr} with ∂S lying above p, p' for the intersection of ℓ_{qr} with ∂S lying above q, and r' for the midpoint of the opposite side of R' from pq. The points p', q' and r' are the best locations to place black points for the purposes of covering points m_s, for small s. However, even their corresponding black discs fail to cover $m_{s'}$, for suitable s'. Specifically, write

$$\begin{aligned} D_p &= D(p', ||p' - q||) = D(p', ||p' - r||), \\ D_q &= D(q', ||q' - p||) = D(q', ||q' - r||), \\ D_r &= D(r', ||r' - p||) = D(r', ||r' - q||). \end{aligned}$$

If the distance of i' from pq is c_i/t, then the heights of D_p and D_q above m_0 are asymptotically $t^3/8c_i$, and D_r only covers m_s for $s < t^3/8c_r$ (asymptotically). However, by construction,

$$c_r \geq \tfrac{3}{2} \max\{c_p, c_q\},$$

so the point $m_{s'}$, for $s' = t^3/7c_r$, will be uncovered by $D_p \cup D_q \cup D_r$. Having identified s', it is straightforward to check that the Voronoi cell V of $\{s', p, q, r\}$ (shown dashed in Figure 1) is entirely contained in $R \cup R'$. Therefore, s' will not be covered by $\mathcal{A}_\lambda$, since, by construction, $V \subset R \cup R'$ is free of black points; all black discs will be stopped by p, q, r, or another red point, before they cover s'. $\square$

Here is a rough intuitive explanation of the proof of Theorem 2. For $i = 0, 1, 2, \ldots$, let us say that a good configuration is of *type* i if the parameter $t = ||p - q||$ satisfies $2^{-(i+1)} \leq t < 2^{-i}$. Close to the threshold, for each i, there will be $O(\lambda^3 n) = o(1)$ good configurations of type i, so that, for fixed i, the probability that a good configuration of type i exists in B_n tends to zero. However, there are $C \log n$ possible types, and so, under the hypotheses of Theorem 2, *some* good configuration will occur in B_n with high probability.

The next theorem shows that if the expected number of good configurations tends to zero, then coverage does in fact occur.

Theorem 3. *If* $\lambda^3 n \log n \to 0$, *then* $p_\lambda(n) \to 1$.

Proof. Suppose that $n \to \infty$ and also that $\lambda^3 n \log n \to 0$. First, we show that we need only worry about coverage of parts of B_n which are close (within distance $\sqrt{8 \log n}$) to a red point. To do this, we tessellate B_n with squares of side length $r = \sqrt{\log n}$. (Some of the squares will not lie entirely inside B_n, but this does not cause problems.) The probability that any small square of the tessellation contains no black point is $e^{-\log n} = n^{-1}$. Since there are $\sim n/\log n$ such squares, the expected number of them containing no black points is asymptotically $1/\log n \to 0$. Consequently, with high probability, every small square contains a black point. Now fix a small square S. If no point of S is within distance $\sqrt{2 \log n}$ of a red point, and if S contains a black point, then all of S will be covered by $\mathcal{A}_\lambda$. Therefore, with high probability, any point of B_n at distance more than $\sqrt{8 \log n}$ from all red points will be covered by $\mathcal{A}_\lambda$, and we may assume this from now on. (Note that we do not condition on the event that every small square contains a black point, as this would affect our estimates. Instead, our arguments below show that when certain conditions are met, coverage occurs, except possibly when some small square contains no black point; however, this last event has probability $o(1)$, so coverage still occurs with high probability.)

It remains to show that the regions of B_n within distance $\sqrt{8 \log n}$ from a red point are covered by $\mathcal{A}_\lambda$. Color such regions yellow. In order to facilitate a division into cases, let us construct a graph $G = G(n, \mathcal{P}')$ on the red points by joining two red points if they lie within distance $R = R(n) = \sqrt{128 \log n}$ of each other. (Such a graph is usually called a *random geometric graph.*) A routine calculation (see for instance [10]) shows that, with high probability, the connected components of G consist of $o(n^{2/3}(\log n)^{-1/3})$ isolated vertices, $o(n^{1/3}(\log n)^{1/3})$ edges, $o(\log n)$ triangles, and $o(\log n)$ paths of length 2 (i.e., paths with 2 edges). This means that each yellow region is associated with either an isolated red vertex, a red edge, a red triangle, or a red path of length 2. We deal with each of these in turn; our argument for triangles will also cover the case for paths of length 2, which we consider as triangles with one "long" edge.

Isolated vertices. Consider the circles of radii $\sqrt{8 \log n}$ and $\sqrt{32 \log n}$ around each isolated red point, and divide the annulus between these circles into 6 equal "sectors", each of area $4\pi \log n$. With high probability, there is a black point inside each sector, and this black point is closer to the isolated vertex than to any other red point. But then the yellow region surrounding the isolated vertex is covered by $\mathcal{A}_\lambda$.

Edges. For a fixed edge $e = pq \in E(G)$, where we may assume $p = (0, 0)$ and $q = (t, 0)$, consider the circles of radii $\sqrt{8 \log n}$ and $\sqrt{32 \log n}$ around p and q. Divide each half-annulus, between two concentric circles and lying outside the "critical strip" $S = [0, t] \times \mathbb{R}$, into 3 equal sectors, each of area $4\pi \log n$. With high probability, there is a black point inside each sector, and this black point is closer to p or q than to any other red point. Thus the yellow regions outside S are covered by $\mathcal{A}_\lambda$. However, coverage of the yellow regions inside the critical strip S is not guaranteed. Indeed, the proof of Theorem 2 shows that such coverage is threatened by the presence of red points at distance $\sim C/t$ from e. G contains edges almost as short as $n^{-1/6}$, so such points may lie almost as far as $n^{1/6}$ from e, almost as much as the typical distance between red points.

We need to show that the edge $e = pq$ is, with high probability, covered from both above and below, so that the yellow regions inside S both above and below e are covered by $\mathcal{A}_\lambda$. It will be sufficient to show that e is covered from above with high probability; an analogous argument will then deal with coverage from below. Let r be the closest point to p, under the condition that the angle rpq is between 0 and π (thus, in this case, r is "above" e), and write $s = ||r - p||$. With notation as in the proof of Theorem 2, the lines ℓ_{pr} and ℓ_{qr} intersect at height $h \geq \frac{s}{2\sqrt{3}}$ above e (see Figure 2 – the worst case is when points p, q and r form an equilateral triangle, because the length of pr

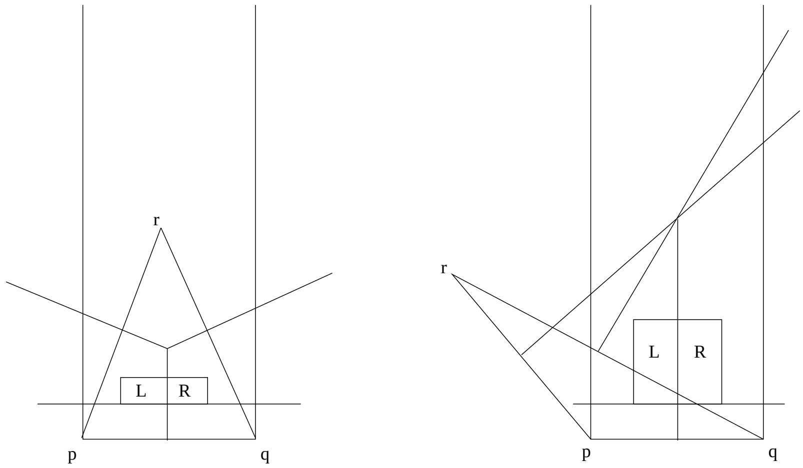

Figure 2: Covering the edge pq from above

is always bigger than that of pq). Now let ℓ be the line parallel to e, lying at height $\sqrt{2\log n}$ above e, and let T be the rectangle with base of length $\frac{t}{2}$ lying on ℓ, of height

$$\frac{h}{2} - \sqrt{2\log n} \geq \frac{s}{4\sqrt{3}} - \sqrt{2\log n} \geq \frac{s}{60},$$

and such that T is bisected by ℓ_{pq}. Every point of T lies below both ℓ_{pr} and ℓ_{qr}, and so is closer to p and q than r. Denoting the left and right halves of T by L and R respectively, we see that if each of L and R contains a black point, then the entire yellow region inside S and above e will be covered by the discs centered at these two points. But, with probability at least $1 - 2e^{-st/240}$, L and R each do contain a black point. Therefore, there exist constants C and C' such that the expected number Y of edges not covered from above can be bounded by

$$\begin{aligned}\mathbb{E}(Y) &\le o(1) + C\lambda n \int_{n^{-1/6}}^{\sqrt{128\log n}} \lambda t \int_{\sqrt{128\log n}}^{\infty} \lambda s e^{-st/240}\, ds\, dt \\ &\le o(1) + C\lambda n \int_{n^{-1/6}}^{\sqrt{128\log n}} \lambda t \int_{0}^{\infty} \lambda x t^{-2} e^{-x/240}\, dx\, dt \\ &= o(1) + C\lambda^3 n \int_{n^{-1/6}}^{\sqrt{128\log n}} t^{-1} \int_{0}^{\infty} x e^{-x/240}\, dx\, dt \\ &= o(1) + C'\lambda^3 n \log n \to 0.\end{aligned}$$

Consequently, with high probability, the yellow regions close to all the edges in G are completely covered by $\mathcal{A}_\lambda$.

Triangles. We expect $o(\log n)$ triangles T in G, and we will classify them by the length x of their smallest sides. In the first case, illustrated in the first two parts of Figure 3, no angle of T is greater than $\frac{9}{10}\pi$. Consider the disc D, centered at the circumcenter c_T of T, of radius $\frac{x}{4}$. If each of the three sectors of D formed from the perpendicular bisectors of the sides of T contains a black point, then the entire interior of T is covered by $\mathcal{A}_\lambda$. (For instance, suppose that the sector corresponding to p contains a black point b; it then follows that the (closure of the) disc centered at b covers p, c_T, and both midpoints m_{pq} and m_{pr} of pq and pr, so that the same closed disc covers the quadrilateral $pm_{pq}c_T m_{pr}$, by convexity. The exterior of T is easily seen to be covered with high probability.) But each of these sectors has area at least $\frac{\pi}{20} \cdot \frac{x^2}{16} = \frac{\pi x^2}{320}$, so that the expected number T_1 of such triangles which are not entirely covered can be bounded by

$$\begin{aligned}\mathbb{E}(T_1) &\le o(1) + C\lambda^2 n \log n \int_{n^{-1/6}}^{\sqrt{128\log n}} \lambda x e^{-\pi x^2/320}\, dx \\ &\le o(1) + C\lambda^3 n \log n \int_{0}^{\infty} x e^{-\pi x^2/320}\, dx \\ &= o(1) + C'\lambda^3 n \log n \to 0,\end{aligned}$$

for some constants C and C'. In the second case, where one angle of T, say the angle at p, is greater than $\frac{9}{10}\pi$, we consider the two rectangles whose centers lie on ℓ_{pq} and ℓ_{pr}, halfway from pq (respectively pr) to the circumcenter of T, whose bases are parallel to the respective sides pq and pr, and whose heights and widths are $\frac{x}{10}$ and $\frac{x}{3}$ respectively (see the third part of Figure 3). If each half of each of these rectangles contains a black point, the interior of T is covered, and so the expected number T_2 of such triangles which are not entirely covered can be bounded by

$$\begin{aligned}\mathbb{E}(T_2) &\leq o(1) + C\lambda^2 n \log n \int_{n^{-1/6}}^{\sqrt{128 \log n}} \lambda x e^{-\pi x^2/60}\, dx \\ &\leq o(1) + C\lambda^3 n \log n \int_0^\infty x e^{-\pi x^2/60}\, dx \\ &= o(1) + C'\lambda^3 n \log n \to 0,\end{aligned}$$

for some constants C and C'. Therefore, with high probability, the interiors of all the triangles in G are covered by $\mathcal{A}_\lambda$, completing the proof of the theorem.

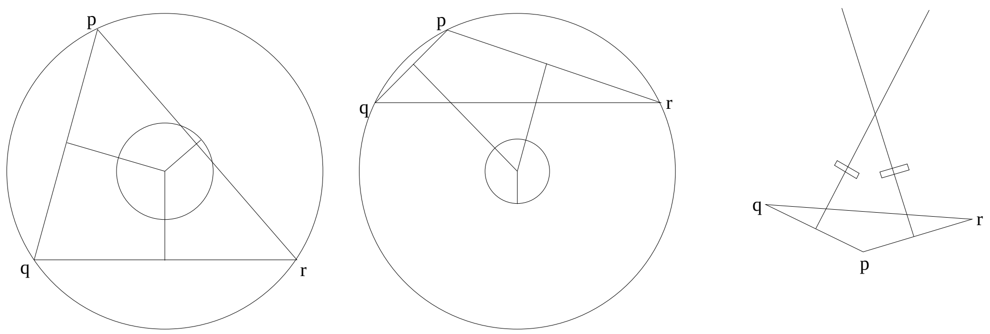

Figure 3: Covering the interior of triangle pqr

□

3 Proof of the main result

More careful estimates, combined with the Stein-Chen method [1], yield the following more precise result, which also shows that good configurations are essentially the only obstructions to coverage.

Theorem 4. *If $\lambda^3 n \log n = y$, then, for sufficiently large n,*

$$e^{-8\pi^2 y} \leq p_\lambda(n) \leq e^{-\frac{2}{3}\pi^2 y}.$$

Proof. Suppose that $\lambda^3 n \log n = y$. We follow the strategy of the proof of Theorem 3, with a few modifications. Define the graph $G = G(n, \mathcal{P}')$ on the red points as in that proof. With high probability, the yellow regions associated with isolated vertices are still covered by black discs, even at this higher range of values of λ.

The first step is to show that the yellow regions inside and close to triangles in G are also still covered, again at this higher range of values of λ, and with high probability. This requires a different argument from before. This time, we overlay each triangle T in G with a tessellation $\mathcal{T}_T$ of side length $(\log n)^{1/5}$ in such a way that $\mathcal{T}_T$ covers T and extends for at least 10 side lengths out from the perimeter of T. Since we only expect $O(\log n)$ triangles in G, and each triangle has area $O(\log n)$, with high probability there will be at most $C(\log n)^{8/5}$ squares in the tessellations, each of which will contain no black points with probability $\exp(-(\log n)^{2/5})$, so, with high probability, every square of every tessellation will contain a black point. Therefore, as in the proof of Theorem 3, all points inside any $\mathcal{T}_T$ not within distance $\sqrt{8}(\log n)^{1/5}$ of a red point are covered by $\mathcal{A}_\lambda$. Accordingly, we color the regions within distance $\sqrt{8}(\log n)^{1/5}$ of such a red point orange.

With this in mind, we now consider a new graph G', whose vertex set consists of all vertices of all the triangles of G. We join two red points if their distance is at most $R' = R'(n) = \sqrt{128}(\log n)^{1/5}$. With high probability, G' contains no triangles, since the expected number of triangles in G' is $C\lambda n \cdot \lambda(\log n)^{2/5} \cdot \lambda(\log n)^{2/5} = C\lambda^3 n(\log n)^{4/5} \to 0$. Each triangle in G splits in G' into either three isolated vertices, or an edge and an isolated vertex. As in the proof of Theorem 3, the orange regions associated with all the "new" isolated vertices are all covered by $\mathcal{A}_\lambda$, with high probability – note that there are only $O(\log n)$ such vertices, and the associated sectors are empty with probability $\exp(-C(\log n)^{2/5})$. Thus we have only to deal with the orange regions associated with edges in G', and indeed only the parts of those orange regions "above and below" such edges, as before.

The expected number of triangles in G with one edge shorter than $1/\sqrt{\log n}$ is $C\lambda n \cdot \lambda(\log n)^{-1} \cdot \lambda(\log n) = C\lambda^3 n \to 0$, so with high probability all edges in G' have lengths longer than $1/\sqrt{\log n}$. Consequently, the expected number Y' of edges of G' whose orange regions are not covered can by bounded, as in the proof of Theorem 3, by

$$\begin{aligned}\mathbb{E}(Y') &\le C\lambda n \int_{(\log n)^{-1/2}}^{\sqrt{128}(\log n)^{1/5}} \lambda t \int_{\sqrt{128}(\log n)^{1/5}}^{\infty} \lambda s e^{-st/240}\, ds\, dt \\ &\le C'\lambda^3 n \log\log n \to 0.\end{aligned}$$

Consequently, with high probability, all the orange regions are covered, so that all the uncovered regions in B_n are associated with *edges* in G.

The detailed strategy for the remainder of the proof is as follows. First, we need to estimate the frequency of uncovered edges (i.e., edges in G whose associated yellow regions are uncovered by black discs). Suppose that this frequency is such that we expect cy uncovered edges in B_n. Then these uncovered edges should be well-approximated by a Poisson process in B_n, and so the probability that there will be no uncovered edges should tend to e^{-cy}. But, following the above remarks, this is also the probability of coverage.

Unfortunately, estimating c itself seems quite hard, since, in contrast to the one-dimensional case, there is no simple necessary and sufficient condition for an edge of G to be covered by $\mathcal{A}_\lambda$. The best we can do is describe a simple necessary condition for coverage (edges *not* satisfying this condition are termed *Type 2 edges*), and a corresponding simple sufficient condition for coverage (edges *not* satisfying such a condition are *Type 1 edges*). Type 2 edges provide a lower bound on c, and hence an upper bound on $p_\lambda(n)$, while Type 1 edges provide an upper bound on c, and a lower bound on $p_\lambda(n)$. To summarize, denoting the sets of Type 1, Type 2, and uncovered edges in B_n by T_1, T_2, and U, we have $T_2 \subset U \subset T_1$. We now turn to the precise descriptions of these types of edge.

Type 1 edges. With reference to Figures 1 and 2, let R be the rectangle whose base is parallel to pq and lies at height $\sqrt{2\log n}$ above pq, whose top is parallel to pq and just touches the lowest of the four intersections of ℓ_{pr} and ℓ_{qr} with S, and whose sides are those of S itself. A sufficient condition for coverage of pq is that pq is *covered from above*, that is, there are sufficiently many black points in R to cover the yellow region above pq; the yellow region below pq is covered, for all such edges in B_n, with high probability. (This condition is not necessary, since black points below pq might by themselves cover the yellow regions on both sides.) We can estimate the number of Type 1 configurations by "projecting" the black points in R to the edge pq, resulting in a one-dimensional process on an interval of length 1 whose intensity is just the area of R, and applying (the proof of) Theorem 1. We will call a rectangle R whose black points do not cover pq from above a *blue rectangle*; Type 1 edges are those associated with blue rectangles.

Type 2 edges. Again with reference to Figures 1 and 2, let V be the dotted Voronoi cell corresponding to a point s', where s' has been chosen to minimize the area of V (and thus maximize the probability that V is free of black points). A necessary condition for coverage of the yellow region above pq is coverage of the point s', and this occurs if and only if a black point lies in V. (This condition is clearly not sufficient.) A rough calculation shows that s', as defined in the proof of Theorem 2, is already almost optimal; the point on ℓ_{pq} which minimizes the area of V is at height asymptotically $t^2/8u$ above pq, where $||p-q|| = t$, and where the circumcenter of pqr is at height u above pq. For this choice of s', V has area approximately tu, and so is free of black points with probability about e^{-tu}. Call a Voronoi cell V without black points a *green triangle*; Type 2 configurations are those associated with green triangles.

When estimating the frequencies of Type 1 and Type 2 edges, we may assume that

$$n^{-1/6}\log n < t < (\log n)^{-1}$$

since edges not satisfying this restriction comprise an asymptotically negligible fraction of both types of edge. Indeed, the expected number of edges of either type with $n^{-1/6} < t < n^{-1/6}\log n$ can be bounded by

$$C\lambda n \int_{n^{-1/6}}^{n^{-1/6}\log n} \lambda t \int_{\sqrt{128\log n}}^{\infty} \lambda s e^{-st/240}\, ds\, dt = C'\lambda^3 n \log\log n \to 0,$$

while the expected number of those with $(\log n)^{-1} < t < \sqrt{128\log n}$ can be bounded by

$$C\lambda n \int_{(\log n)^{-1}}^{\sqrt{128\log n}} \lambda t \int_{\sqrt{128\log n}}^{\infty} \lambda s e^{-st/240}\, ds\, dt = C''\lambda^3 n \log\log n \to 0.$$

The next step is to show that the edges of both types are well-approximated by Poisson processes, so that, in particular, if we expect $c_1 y$ edges of Type 1, and $c_2 y$ edges of Type 2, we will have $e^{-c_1 y} \le p_\lambda(n) \le e^{-c_2 y}$. For this we use the Stein-Chen method [1], in the following form (taken from [3]).

Proposition 5. *Let $\xi_1, \xi_2, \ldots$ be a countable collection of independent random variables, and let $Z_1, Z_2, \ldots$ be a countable collection of Bernoulli random variables, where Z_i is a function of the values of ξ_j, $j \in S_i$. Suppose that $\sum_i \mathbb{E}(Z_i) = \mu$, and let*

$$b_1 = \sum_{i,j\,:\; S_i \cap S_j \neq \emptyset} \mathbb{E}(Z_i)\mathbb{E}(Z_j),$$

$$b_2 = \sum_{i,j\,:\; S_i \cap S_j \neq \emptyset,\, i \neq j} \mathbb{E}(Z_i Z_j)$$

and $Z = \sum_i Z_i$. Then for all r,

$$|\,\mathbb{P}(Z = r) - e^{-\mu}\mu^r/r!\,| \leq \frac{1 - e^{-\mu}}{\mu}(b_1 + b_2).$$

(Theorem 1 in [1] includes another term b_3 that bounds dependency when $S_i \cap S_j = \emptyset$, but in our case, as in [3], $b_3 = 0$.)

To make the collection of events countable, we proceed as in [3], dividing B_n up into a very fine grid, and moving all points of $\mathcal{P}$ and $\mathcal{P}'$ to their nearest grid point. For fixed n and y, we can make the number of both types of edge in this discrete version equal to the number in the original with probability arbitrarily close to 1. The random variables ξ_i record whether or not the i^{th} grid point is occupied for each of $\mathcal{P}$ and $\mathcal{P}'$, and, for every grid point i, we introduce a variable Z_i indicating whether a Type 1 (or Type 2) edge exists with its midpoint at i. The set S_i can be taken to be the set of grid points within distance $n^{1/6}(\log n)^{-1/2}$ of i.

The first thing to check is that, with high probability, the Z_i really are Bernoulli random variables. For a pair of, say, Type 1 configurations to share a common edge, we require four red points p, q, r and s, such that q is within distance t of p, and then both r and s are within distance $1/t$ of p, for t in the range specified above. But the expected number of such configurations is certainly at most $C\lambda^3 n \log n \cdot \lambda n^{1/3} \to 0$.

We are interested in the case where μ is a positive constant, so we must show that b_1 and b_2 tend to zero as $n \to \infty$. First, we have

$$b_1 = \sum_i \mathbb{E}(Z_i) \sum_{j:S_i \cap S_j \neq \emptyset} \mathbb{E}(Z_j) \leq C\mu \cdot \mu \frac{n}{n^{1/3}(\log n)^{-1}} \to 0.$$

Second, for some pair Z_i and Z_j with $S_i \cap S_j \neq \emptyset$ to both equal 1, we require the presence of five red points p, p', q, q' and r, all within distance $n^{1/6}$ of each other, such that p and q are also within distance t, and r is within distance $1/t$ of p, for t in the range specified above. The expected number of such configurations is certainly at most $C\lambda^3 n \log n \cdot \lambda n^{1/3} \cdot \lambda n^{1/3} \to 0$, so $b_2 \to 0$ also.

It remains to calculate c_1 and c_2. Suppose that the circumcenter of triangle pqr lies at height between u and $u + du$ above pq. This means that r must lie in an asymmetrical annulus of area $2\pi u\, du$. (Note that we have overcounted the number of edges pq by a factor of 2, which we correct for by assuming that u is "above" pq.) Under these circumstances, the rectangle R has area $(1 + o(1))ut$, and will be blue with probability asymptotically $(1+ut/2)e^{-ut/2}$, while the Voronoi cell V also has area $(1+o(1))ut$, and will be green with probability asymptotically e^{-ut}. (All asymptotic statements and notation are as $n \to \infty$.) Consequently, making the substitution $x = ut$ in both integrals,

$$\begin{aligned}
\mathbb{E}(|T_1|) &\sim \lambda n \int_{n^{-1/6}\log n}^{(\log n)^{-1}} 2\pi\lambda t \int_{\sqrt{128\log n}}^{\infty} 2\pi\lambda u \left(1 + \frac{ut}{2}\right) e^{-ut/2}\, du\, dt \\
&= 4\pi^2\lambda^3 n \int_{n^{-1/6}\log n}^{(\log n)^{-1}} \frac{1}{t}\, dt \int_{t\sqrt{128\log n}}^{\infty} x\left(1+\frac{x}{2}\right) e^{-x/2}\, dx \\
&\sim 4\pi^2\lambda^3 n \int_{n^{-1/6}\log n}^{(\log n)^{-1}} \frac{1}{t}\, dt \int_0^{\infty} x\left(1+\frac{x}{2}\right) e^{-x/2}\, dx \\
&\sim 8\pi^2\lambda^3 n \log n = 8\pi^2 y,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}(|T_2|) &\sim \lambda n \int_{n^{-1/6}\log n}^{(\log n)^{-1}} 2\pi\lambda t \int_{\sqrt{128\log n}}^{\infty} 2\pi\lambda u e^{-ut}\, du\, dt \\
&= 4\pi^2\lambda^3 n \int_{n^{-1/6}\log n}^{(\log n)^{-1}} \frac{1}{t}\, dt \int_{t\sqrt{128\log n}}^{\infty} x e^{-x}\, dx \\
&\sim 4\pi^2\lambda^3 n \int_{n^{-1/6}\log n}^{(\log n)^{-1}} \frac{1}{t}\, dt \int_0^{\infty} x e^{-x}\, dx \\
&\sim \frac{2}{3}\pi^2\lambda^3 n \log n = \frac{2}{3}\pi^2 y,
\end{aligned}$$

completing the argument. □

The variable x in the above calculation can be interpreted as the amount by which a "generic" configuration has been "stretched"; the frequency of blue rectangles and green triangles corresponding to a fixed value of t and with $x \leq ut \leq x + dx$ is exponentially decreasing in x.

As explained above, it does seem likely that there exists a single constant c such that if $\lambda^3 n \log n = y$ then $p_\lambda(n) \to e^{-cy}$. It might even be possible to provide an explicit expression for c. Finally, it would be interesting to investigate the problem in higher dimensions.

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